Econ 210B
Discussion Session N°9

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UC San Diego

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Housekeeping

We are almost done with 210B! (and with the second quarter of first year!)

Week	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Week 9	03-03 L15	03-04	03-05 L16	03-06 DS9	03-07	03-08	03-09
Week 10	03-10 L17	03-11	03-12 L18	03-13 DS10	03-14	03-15	03-16 PS4
Week 11	03-17 220B Final	03-18 200B Final	03-19 **210B Final**	03-20	03-21	03-22	03-23

- 2 more lectures on Search and Matching
- Final Exam: 8 am, Wednesday 3/19

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Remaining discussion sessions

- Today 3/6: OLG
- Next Thursday 3/13: Search and Matching
- Review Session

  Friday 3/14 ?
  Tuesday 3/18 ?

Plan for Today

OLG Review

PS4 Problem D: OLG Production Economy with Capital-Dependent Technology

Final 2024 Problem B: OLG Production Economy

OLG Review: Set Up

generation \ time	1	2	3	4	
0	$u(c_1^0), y_1^0$				
1	$u(c_1^1), y_1^1$	$u(c_2^1), y_2^1$			
2		$u(c_2^2), y_2^2$	$u(c_3^2), y_3^2$		
3			$u(c_3^3), y_3^3$	$u(c_4^3), y_4^3$	
4				$u(c_4^4), y_4^4$	

in endowment economies, y_t^i is exogenous process

in production economies, y_t^i is endogenous labor and/or capital income

- Time: $t = 1, 2, 3, \ldots$
- Generations: $i = 0, 1, 2, \ldots$
- Preferences: Generation $i \geq 1$, at $t = i$: $u(c_t^i) + u(c_{t+1}^i)$

  Generation $i = 0$ at $t = 1$: $u(c_1^0)$

OLG Review: Endowment Economy

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OLG Review: Endowment Economy

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- Endowments: $t = i$: $y_t^i \geq 0,\ y_{t+1}^i \geq 0$
  $t \neq i$: $y_t^i = 0$

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- CE: Set $\{q_t^0\}_{t=0}^{\infty}$, $\{c_t^i\}_{t=1}^{\infty}$, $i = 0, 1, 2, \ldots$, such that:

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  1. Households Optimize

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$$\text{s.t.} \quad \sum_{t=1}^{\infty} c_t^i q_t^0 \leq \sum_{t=1}^{\infty} y_t^i q_t^0$$

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$$\text{FOC} \quad i = t : \quad \{c_t^i\} \qquad u'(c_t^i) = \mu^i q_t^0$$

$$\{c_{t+1}^i\} \quad u'(c_{t+1}^i) = \mu^i q_{t+1}^0$$

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$$i \neq t : \quad \{c_t^i\} \quad c_t^i = 0$$

  2. Markets clear

$$\sum_{i=1}^{N} c_i^t = \sum_{i=1}^{N} y_i^t \quad \forall t \implies c_t^i + c_t^{i-1} \leq y_t^i + y_t^{i-1} \quad \forall t$$

OLG Review: Stationary Equilibria in Endowment Economy

- Assume endowment process: $y_i^i = 1 - \varepsilon,\ y_{i+1}^i = \varepsilon,\ \varepsilon \in (0, \frac{1}{2})$

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  1. H-Equilibrium: High interest rate ($R = 1$)
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  2. L-Equilibrium: Low interest rate ($R < 1$)
     - AD prices: $q_t^0 = 1$
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     From FOC

$$\frac{1}{R} = \frac{q_{t+1}^0}{q_t^0} = \frac{u'(\varepsilon)}{u'(1-\varepsilon)} > 1$$

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- Result: H PD L (1st Welfare Theorem fails)

OLG Review: Offer Curve

Offer curve: locus of (c_t^t, c_{t+1}^t) that solves:

$$\max_{c_t^i, c_{t+1}^i} \quad u(c_t^i) + u(c_{t+1}^i)$$
$$\text{s.t.} \quad c_t^i + \alpha_t c_{t+1}^i \leq y_i^t + \alpha_t y_{t+1}^i$$

Note $\alpha_t = \frac{q_{t+1}^{\mathbf{0}}}{q_t^{\mathbf{0}}}$ and $i = t$.

From the FOC we get:

$$\alpha_t = \frac{u'(c_{t+1}^i)}{u'(c_t^i)}$$

Plugging in the BC:

$$c_t^i + \frac{u'(c_{t+1}^i)}{u'(c_t^i)} c_{t+1}^i = y_t^i + \frac{u'(c_{t+1}^i)}{u'(c_t^i)} y_{t+1}^i$$

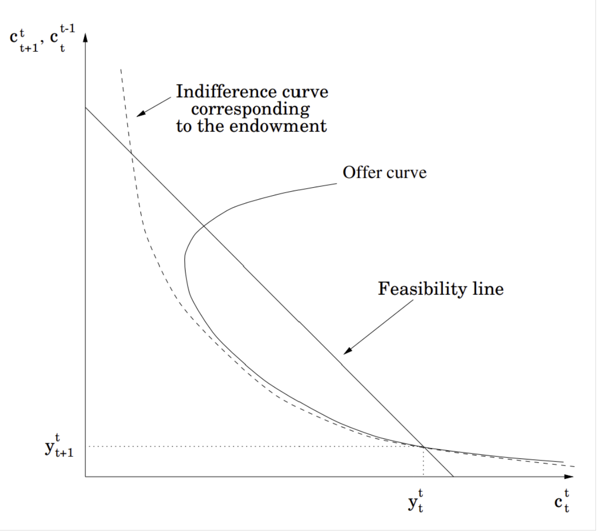
We denote the offer curve by the implicit function

$$\psi(c_t^t, c_{t+1}^t) = 0$$

OLG Review: Offer Curve

- The idea is that the offer curve contain all the "best combinations" of consumption today and tomorrow. So if we have an equilibrium it has to in the offer curve.
- To represent it graphically, note:
  - Given one α, the offer curve is the point of tangency between the budget constraint and the highest indifference curve.
  - If we chose different α, we construct the entire curve. Clearly, the curve will depend on the endowment vector.
- Note that if we consider the indifference curve that goes through the endowment vector, the offer curve has to lie above it at all times. Any combination (c_t^t, c_{t+1}^t) different that endowment on the supply curve has to yield higher utility by construction (at any price, consuming endowments is always a possibility)
- By construction, the following property is also true: at the intersection between the offer curve and a straight line through the endowment point, the straight line is tangent to an indifference curve.

OLG Review: Offer Curve



$$\text{Offer curve:} \quad \psi(c^t_t, c^t_{t+1}) = 0$$

$$\text{Feasibility:} \quad c^i_t + c^{i-1}_t = y^i_t + y^{i-1}_t$$

OLG Review: Offer Curve

We can use the offer curve and a straight line depicting feasibility in the (c_t^t, c_t^{t-1}) plane to build a procedure finding all equilibrium allocations and prices.

In particular, we can use the following pair of difference equations to solve for an equilibrium allocation. For $i \geq 1$, the equations are

$$\psi(c_t^t, c_{t+1}^t) = 0 \quad ; \quad c_t^t + c_t^{t-1} = y_t^t + y_t^{t-1}$$

How to get an equilibrium from the diagram:

(1) Chose initial c_1^1 to the right of H equilibrium

(2) From offer curve get c_2^1

(3) From feasibility obtain c_2^2

(4) Repeat from step (1) using c_2^2

See Luguisvst and Sargent (Chapter 9) for a more detailed explanation and examples of how to construct equilibria.

OLG Review: Offer Curve

Important result:

1. All equilibria except one converge to L
2. The equilibrium not converging is H
3. In all non-stationary equilibria, $c^i_t > c^i_{t+1}$, for $t = i$, so

$$\begin{aligned}\alpha_t &= \frac{u'(c^i_{t+1})}{u'(c^i_t)} > 1 \\ &= \frac{q^0_{t+1}}{q^0_t} > 1 \\ \implies &\lim_{t\to\infty} q^0_t = \infty\end{aligned}$$

PS4 Problem D: OLG Production Economy with Capital-Dependent Technology

Consider a production economy with overlapping generations of two-period-lived agents.

Agent Preferences: Young agents (born on t) supply labor inelastically when young at competitive wage w_t and consume a single good.

$$\ln(c_t^t) + \beta \ln(c_{t+1}^t), \quad \beta \in (0, 1)$$

Young agents can save by purchasing capital and renting it out to firms the following period at gross rate R_{t+1}. Superscript denotes generation's time birth, subscript denotes period of time.

Production: A representative firm rents labor and capital, maximizing per-period profits:

$$y_t = F(k_t, 1) = f(k_t)$$

where

$$f(k) = \begin{cases} k^\alpha, & \text{if } k < k_A \\ Ak^\alpha, & \text{if } k \geq k_A \end{cases}, \quad \alpha \in (0, 1), A > 1$$

PS4 Problem D: OLG Production Economy with Capital-Dependent Technology

(a) Maximization Problem: Write the maximization problem for generation t in this economy. Derive all optimality conditions. Obtain an expression for optimal savings of generation t as a function of income at t.

(b) Competitive Equilibrium: Define equilibrium and derive equilibrium law of motion for capital.

(c) Steady-State Capital: Investigate the existence of two steady states and explain their intuition.

(d) Convergence to High Capital: Analyze if the economy converges to $k^* > k_A$ from $k_0 < k_A$.

(e) Dynamic Inefficiency: Determine conditions under which equilibrium is dynamically inefficient.

(f) Open Economy: Characterize steady-state capital in an open economy with risk-free assets.

(a) Maximization Problem

- Derive the optimal saving strategy of generation t.
- Obtain all the optimality conditions.

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The problem for generation t is

$$\begin{aligned} \max_{c_t^t, c_{t+1}^t, s_t} \quad & \ln(c_t^t) + \beta \ln(c_{t+1}^t) \\ \text{s.t.} \quad & c_t^t + s_t \leq w_t \\ & c_{t+1}^t \leq R_{t+1} s_t \end{aligned}$$

Note that agents do not derive any utility from leisure therefore they always work the entire unit of time.
The Lagrangian is

$$\mathcal{L} = ln(c_t^t) + \beta ln(c_{t+1}^t) + \lambda_t(w_t - c_t^t - s_t) + \lambda_{t+1}(R_{t+1}s_t - c_{t+1}^t)$$

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The optimality conditions are:

$$\{c_t^t\} \quad \frac{1}{c_t^t} - \lambda_t = 0$$

$$\{c_{t+1}^t\} \quad \frac{\beta}{c_{t+1}^t} - \lambda_{t+1} = 0$$

$$\{s_t\} \quad -\lambda_t + \lambda_{t+1}R_{t+1} = 0$$

$$\{CS\} \quad \lambda_t(w_t - c_t^t - s_t) = 0 \quad ; \quad \lambda_t \geq 0$$

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\end{aligned}$$

By combining the optimality conditions for consumption and savings we can see that

$$\frac{1}{c_t^t} = \frac{\beta R_{t+1}}{c_{t+1}^t}$$

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By combining the optimality conditions for consumption and savings we can see that

$$\frac{1}{c_t^t} = \frac{\beta R_{t+1}}{c_{t+1}^t}$$

Also, since both restrictions will be satisfied with equality, we can replace consumption to obtain the expression

$$\frac{1}{w_t - s_t} = \frac{\beta}{s_t} \Longrightarrow s_t = \frac{\beta}{1+\beta} w_t$$

(b) Competitive Equilibrium

- Define a competitive equilibrium including market clearing conditions.

- Derive an expression for the law of motion for capital.

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  Equilibrium is defined as: (1) a set of prices w_t and R_t; (2) consumption allocations $\{c_t^t, c_{t+1}^t\}$ savings s_t and labor choices n_t; (3) and aggregate labor N_t and capital K_t such that:

  (i) (2) solves the problem for each generation given (1);

  (ii) (3) solves the problem for the firm given (1);

  (iii) Market Clearing:
    - Goods: $c_t^t + c_t^{t+1} = f(k_t) - k_{t+1}$
    - Assets: $K_{t+1} = s_t$
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- Derive an expression for the law of motion for capital.

  Market for factors are competitive $\implies$ $R_t = f'(k_t)$, $w_t = f(k_t) - f'(k_t)k_t$.

$$k_{t+1} = s_t = \frac{\beta}{1+\beta}[f(k_t) - f'(k_t)k_t]$$

  And using $f(k)$:

$$k_{t+1} = \begin{cases} \frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha & \text{if } k_t < k_A, \\ \frac{\beta}{1+\beta}(1-\alpha)Ak_t^\alpha & \text{if } k_t \geq k_A \end{cases}$$

(b) Law of Motion for Capital

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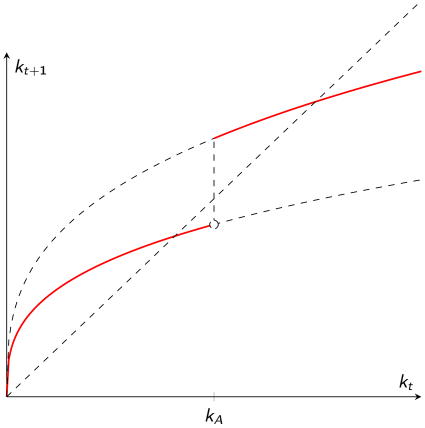

45 degree line: points capital tomorrow will be the same as capital today.

Review: Law of Motion for Capital

$$\text{From FOC:} \quad u'(w_t - s_t^i) = \beta R_{t+1} u'(R_{t+1} s_t^i)$$
$$\text{Demand for savings:} \quad k_{t+1} = s(w_t, R_{t+1})$$

$\implies$ Implicit function for law of motion for capital:

$$\boxed{\frac{dk_{t+1}}{dk_t} = -\frac{k_t f''(k_t) s_w}{1 - s_R f''(k_{t+1})}}$$

To get there:

- Substitute w_t from labor demand $w_t = f(k_t) - f'(k_t)k_t$
- Substitute R_{t+1} from capital demand $R_{t+1} = f'(k_{t+1})$
- Total differentiation

Dynamics of k depend on:

- f'': Concavity $f'' < 0$ vs convexity $f'' > 0$ of f
- s_R: Impact of R on savings demand, which depends of what effect dominates (income $s_R < 0$ vs. substitution $s_R > 0$)
- s_w: Impact of w on savings demand. We assume $s_w \in (0, 1)$.

Example: Given $f'' < 0$, $s_w > 0$:

$$\frac{dk_{t+1}}{dk_t} > 0 \Rightarrow k_t \text{ increasing over time}$$

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  If k_A generates two ss, we have two production functions (differing in TFP), each with the possibility of its own ss.

  Multiple equilibria and poverty traps when $f(k)$ has increasing returns for some k, $f''(k) > 0$.

(d) Convergence to High Capital Steady State

- Analyze whether the economy converges to $k^* > k_A$ when starting at $k_0 < k_A$.
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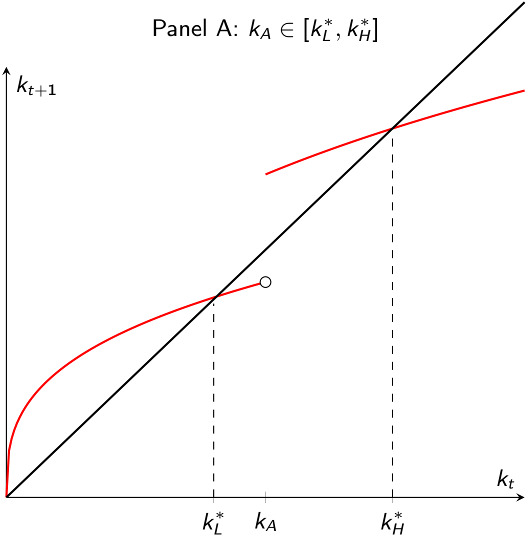

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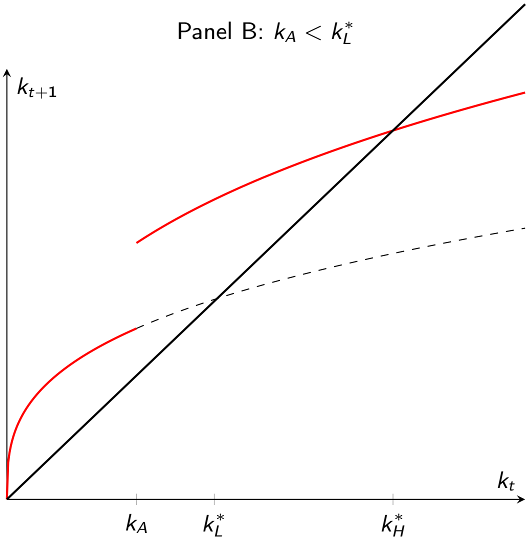
Panel B: $k_A < k_L^*$



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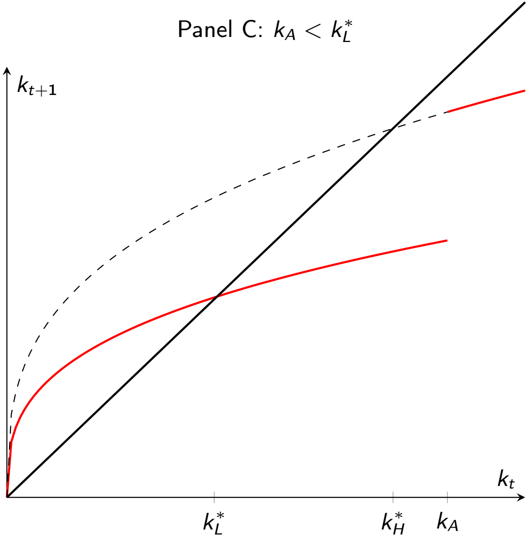
Panel C: $k_A < k_L^*$



(d) Convergence to High Capital Steady State

Concluding:

- We can see from the graphs that if $k_A < k_L^*$ the economy will converge to $k_H^* > k_A$ regardless of where we start.
- However, in the other two cases we will converge to $k_L^* < k_A$.
- While the case with only one equilibrium is trivial, for the case with two equilibrium this will be true only for $k_0 < k_A$.

(e) Dynamic Inefficiency

- Determine conditions under which the equilibrium is dynamically inefficient.

- Explain the source of inefficiency and reconcile capital accumulation with productivity improvements.

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- Determine conditions under which the equilibrium is dynamically inefficient.

  Let $c^* = f(k^*) - k^*$. Then we can use the same analysis we saw in class to see that:

  $dc_t = -dk^* > 0$ and $dc_{t+1} = (f'(k^*) - 1)dk^*$.
  So dynamic inefficiency requires $f'(k^*) < 1$.
  Since $f'(k) = \alpha A_t k^{\alpha-1}$ the previous condition is equal to $k^* > (\alpha A_t)^{\frac{1}{1-\alpha}}$ where A_t is either A or 1 depending which part of the production function we are considering.

  In any case, this gives a dynamically inefficient equilibrium if

$$\left[\frac{\beta}{1+\beta}(1-\alpha)A_t\right]^{\frac{1}{1-\alpha}} > (\alpha A_t)^{\frac{1}{1-\alpha}} \implies \frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}$$

- Explain the source of inefficiency and reconcile capital accumulation with productivity improvements.

Review: Dynamic Inefficiency

Dynamic inefficiency: Reducing capital today does not reduce aggregate consumption in any future date or event, increasing it in some.

- In steady state:
$$c^* = f(k^*) - k^*$$
- Suppose we reduce savings by $dk < 0$ to increase consumption $dc_t = -dk > 0$.
- The steady state would be dynamically inefficient if $\forall j > 0 \; dc_{t+j} > 0$

$$dc_{t+j} = f'(k^*)dk - dk > 0$$
$$\iff f'(k^*)dk > dk$$
$$f'(k^*) < 1$$

- Note $f'(k^G) = 1$:

$$k^* > k^G \iff \text{Economy is dynamically inefficient}$$

(f) Open Economy and Financial Markets

- Consider an open economy with access to risk-free assets at gross return $\hat{R}$. Borrowing is not allowed, and the economy is small so it has no impact on $\hat{R}$.

- Characterize steady-state capital as a function of $\hat{R}$.

- Discuss whether financial development (opening the economy to international financial markets) promotes sustained economic growth.

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$$f'(k) = \hat{R}$$

  As before, depending on where k_A is located, we could have that one or two ss could be possible.
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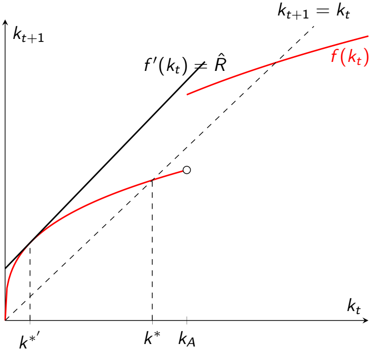
  New ss could be either lower or higher than before.
  If we started in the low equilibrium and had access to higher external interest rates, internal capital would actually decrease.

(f) Open Economy and Financial Markets

- Discuss whether financial development (opening the economy to international financial markets) promotes sustained economic growth.

  If we started in the low equilibrium and had access to higher external interest rates, internal capital would decrease until $f'(k^{*'}) = \hat{R}$.
  In the graph, it would move from k^* (where $k_{t+1} = k_t$), to $k^{*'}$.



Final 2024 Problem B: OLG Production Economy

- OLG economy where individuals live for three periods: young, adult, and old.
- When young and adult individuals are endowed with one unit of time they supply (labor) inelastically.
- The size of each generation is normalized to 1 and constant, so the total population is 3.
- Individuals can save by holding capital or lend/borrow via a competitive credit market.
  - Young individuals can borrow up to a fraction $\phi \in [0, 1]$ of the present value of their labor income as adults.
  - Adults and old individuals cannot borrow.
- $u(c) = \ln c$, $\beta = 1$

Final 2024 Problem B: OLG Production Economy

- Output in the economy is produced by a representative firm with a constant returns to scale technology:

$$Y = K^{\alpha} N^{1-\alpha},$$

where K is aggregate capital, and N is total labor supplied.

- Markets for factors are perfectly competitive.
- The ownership of initial capital K_0 is equally split between initial adults and initial old.

(a) Write the maximization problem for generation t in this economy. Derive all the optimality conditions. Define a competitive equilibrium for the economy, making sure to include all the relevant market clearing conditions.

(b) Without solving fully for the equilibrium, can you tell from the optimality conditions and the market clearing conditions whether the young individuals will ever borrow in equilibrium? If your answer depends on parameters, what are the key parameters that determine whether borrowing will take place? Please discuss.

(a) Maximization Problem

Define the maximization problem for generation t:

- Derive all optimality conditions.
- Define a competitive equilibrium including all market clearing conditions.

(b) Borrowing Decision in Equilibrium

- Can young individuals borrow in equilibrium?
- Key parameters influencing borrowing behavior:
  - Interest rate
  - Wage profile over life cycle
  - Capital return
  - Borrowing constraint ϕ