Econ 210B
Discussion Session N°8

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Plan for Today

Aiyagari Diagram

Hugget Model

PS3 Problem D: Aiyagari Model with Occupational Choice (Final 2024)

Stationary Recursive Competitive Equilibrium (SRCE)

Note

- Asset demand A' becomes capital supply next period $(K^s)'$
- In the SRCE $K = K'$, $A = A'$

  From market clearing:

$$\begin{array}{rl} \text{Capital:} & K^d = K^s \\ \text{Assets:} & A' = (K^s)' \end{array} \quad \Rightarrow \quad K = A$$

Solution for the SRCE is finding λ^* and r^* s.t.:

$$\lambda^* = \mathcal{H}(\lambda^*) \tag{1}$$

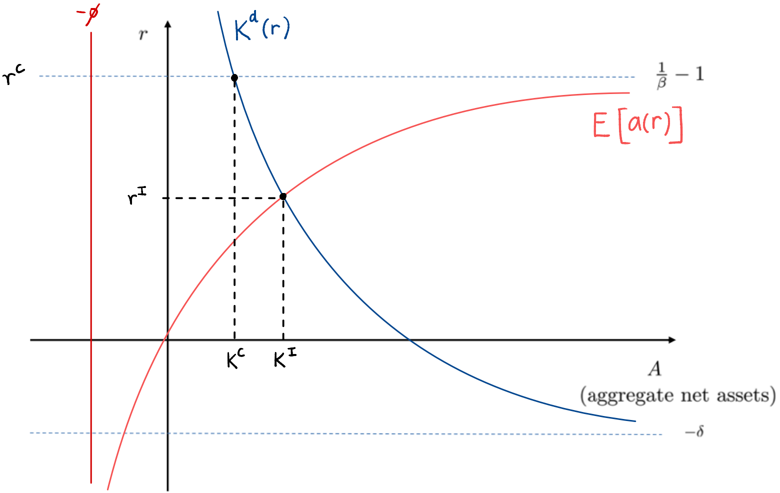
$$K(r^*) = A(r^*; \lambda^*) \tag{2}$$

Aiyagari Diagram

$K^d(r)$: $r = F_K - \delta$

$E[a(r)] = \sum_s \sum_a \lambda(a,s)\, g(a,s)$ $(= A')$

borrowing limit: $a' \geq -\phi$



Practice Question

Using the equilibrium of $K(r)$ and $A(r)$

1. Show how equilibrium capital and interest rate would react to a stricter borrowing limit $-\phi' > -\phi$
2. Characterize equilibrium capital and interest rate when $-\phi \to -\infty$
3. Explain how the result of 2. relates to the complete markets allocation

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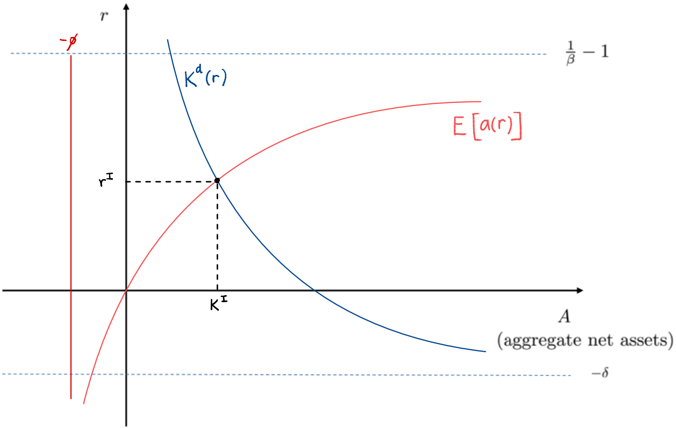

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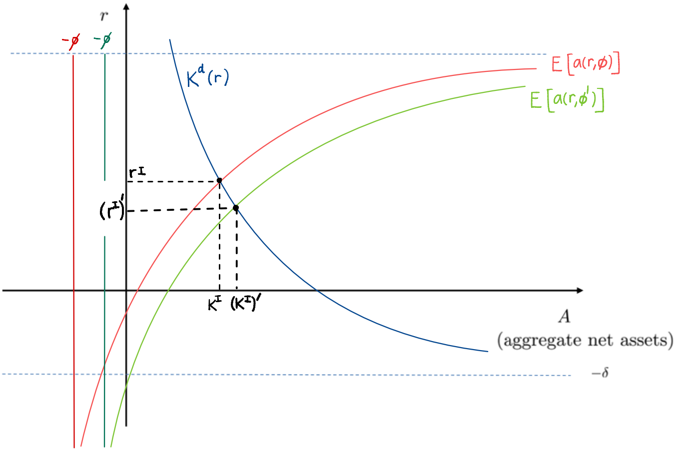

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2. Characterize equilibrium capital and interest rate when $-\phi \to -\infty$

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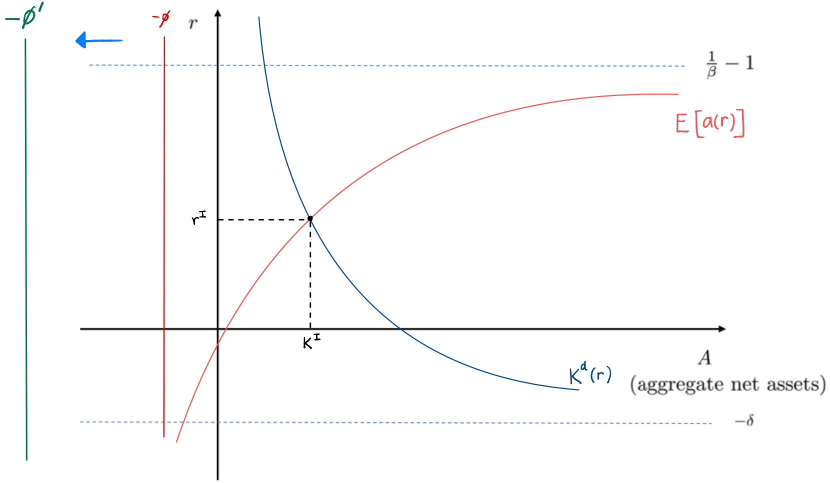

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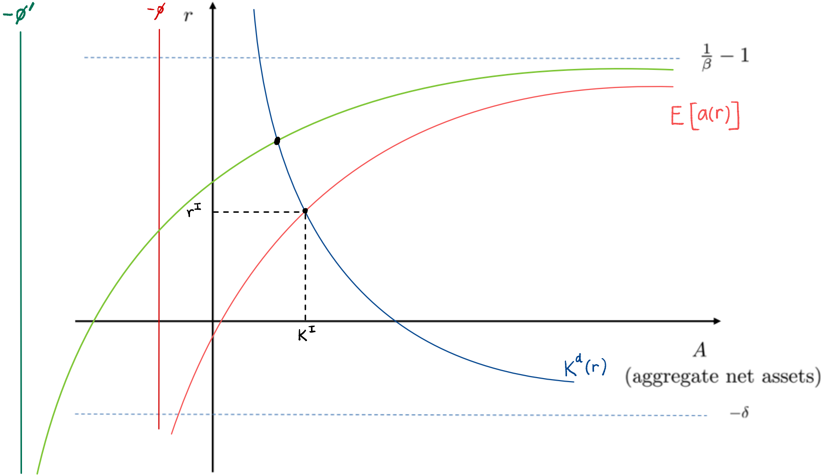

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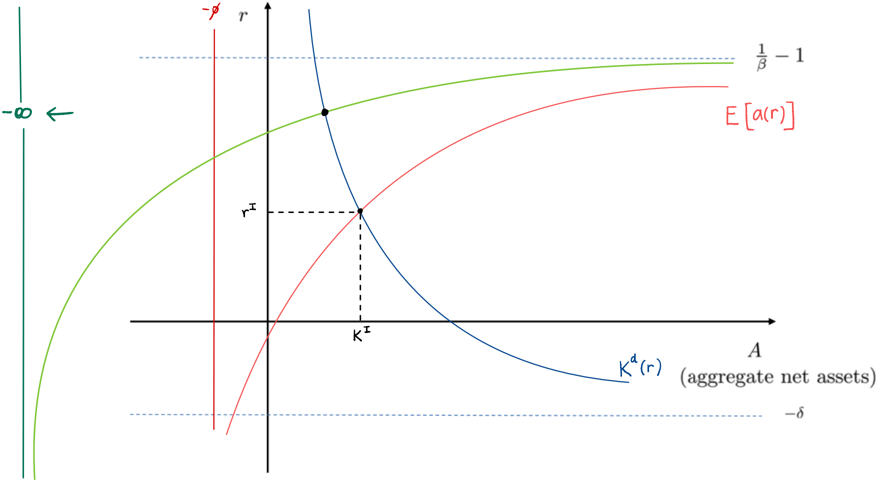

Practice Question

Note $r^c = \frac{1}{\beta} - 1$ ★ is the steady state interest rate in the neoclassical model.

(1) $F_K = r + \delta$

(always holds)

(2) $F_K(K^{ss}) = \frac{1}{\beta} - (1-\delta)$

(condition for SS)

(1) and (2) give you ★

2. Characterize equilibrium capital and interest rate when $-\phi \to -\infty$

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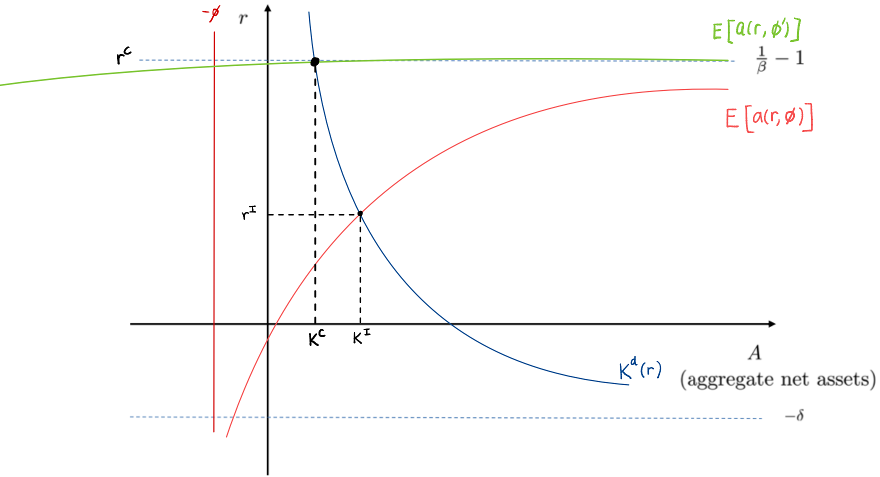

Aiyagari Model with no Capital (Huggett Model)

$K^d(r)$: $K = 0$

$E[a(r)] = \sum_s \sum_a \lambda(a,s) g(a,s)$ $(= A')$

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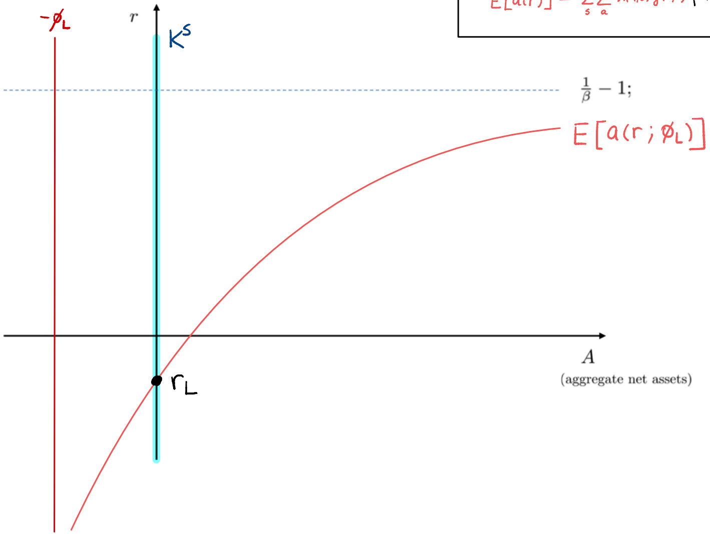

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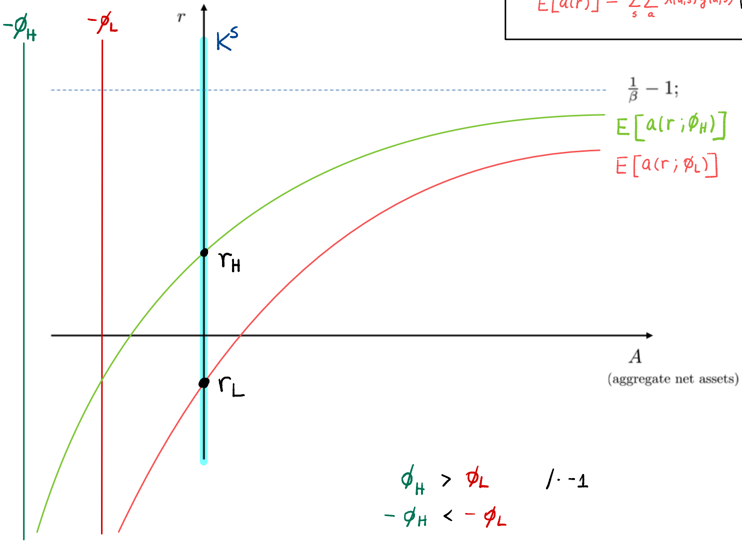

$\phi_H > \phi_L \quad / \cdot -1$

$-\phi_H < -\phi_L$

PS3 Problem D: Aiyagari Model with Occupational Choice

Consider a modified version of the Aiyagari model where individuals have an occupation choice.
Individuals face idiosyncratic labor productivity shocks $s \in \{s_L, s_H\}$ and maximize the expected lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t))\pi(s^t).$$

They earn income either by working at wage w or operating a technology with decreasing returns to scale:

$$\epsilon f(k, n),$$

where k is capital and n is labor.

PS3 D: Occupational Choice and Value Functions

- Each individual has an entrepreneurial ability ϵ drawn from a uniform distribution on $[0, \bar{\epsilon}]$.
- The value function of a worker is $W(a, s, \epsilon)$, while that of an entrepreneur is $V(a, s, \epsilon)$.
- The occupational choice variable $x \in \{0, 1\}$ determines their status:
  - $x = 0$: Worker
  - $x = 1$: Entrepreneur

  The occupation choice is made one period in advance.
- The budget constraints are:

$$c + a' = \pi(a, s, \epsilon), \quad \text{(Entrepreneur)} \tag{1}$$
$$c + a' = ws + (1 + r)a. \quad \text{(Worker)} \tag{2}$$

We assume $a' > 0$

PS3 D: Aiyagari Model with Occupational Choice

(a) Write the maximization problem in recursive form for an individual who is a worker at time t, and for an individual who is an entrepreneur at time t.

(b) The profits for an entrepreneur can be written as

$$\pi(a, s, \epsilon) = \max_{k,n} \{\epsilon f(k, n) + (1 - \delta)k - (1 + r)(k - a) - w \max\{n - s, 0\}\}$$

subject to the borrowing constraint:

$$k - a \leq \gamma a, \quad \gamma \in [0, 1].$$

(i) What factors limit the borrowing capacity of the entrepreneur?
(ii) How would you explain the form taken by the cost of factors in the profit function above?
(iii) Who owns capital in the problem, and how can you make that determination?

(c) Go as far as you can in characterizing the solution to the Aiyagari model with occupational choice in a stationary recursive competitive equilibrium.

PS3 D (a): Recursive Maximization Problem

Worker:

$$W(a,s,\epsilon) = \max_{\{c,a',x\}} \left\{ u(c) + \beta \left((1-x)\mathbb{E}[W(a',s',\epsilon)] + x\mathbb{E}[V(a',s',\epsilon)] \right) \right\}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &= ws + (1+r)a, \\ a' &\geq 0, \\ x &\in \{0,1\}. \end{aligned}$$

Entrepreneur:

$$V(a,s,\epsilon) = \max_{\{c,a',x\}} \left\{ u(c) + \beta \left(x\mathbb{E}[V(a',s',\epsilon)] + (1-x)\mathbb{E}[W(a',s',\epsilon)] \right) \right\}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &= \pi(a,s,\epsilon), \\ a' &\geq 0, \\ x &\in \{0,1\}. \end{aligned}$$

PS3 D: (b) Entrepreneurial Profits

Profits for an entrepreneur are given by:

$$\pi(a, s, \epsilon) = \max_{k,n} \{\epsilon f(k, n) + (1 - \delta)k - (1 + r)(k - a) - w \max\{n - s, 0\}\}$$

$$\text{s.t.} \quad k - a \leq \gamma a, \quad \gamma \in [0, 1].$$

(i) What factors limit borrowing capacity?

- $(k - a)$: Excess capital (amount of capital the entrepreneur must borrow in capital/financial markets) is bounded above by the fraction γ of own wealth a.
- γ: Collateral factor
- a: Own wealth

PS3 D: (b) Entrepreneurial Profits

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(ii) Explain costs form

$$c(n, k) = \underbrace{(1 + r)(k - a)}_{\text{Capital cost}} + \underbrace{w \max\{n - s, 0\}}_{\text{Labor cost}}$$

- For each unit of capital borrowed in the capital market, the entrepreneur must pay $(1 + r)$.
- If the entrepreneur's choice of labor is higher than her own efficiency labor units (i.e., $n > s$), she must employ the difference $(n - s)$ in labor markets and pay a wage rate w for each unit.

PS3 D: (b) Entrepreneurial Profits

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(iii) Who owns capital? Why?

$$c(n, k) = \underbrace{(1 + r)(k - a)}_{\text{Capital cost}} + \underbrace{w \max\{n - s, 0\}}_{\text{Labor cost}}$$

- Some capital is owned by the entrepreneur: a.
- Some capital is purchased at capital markets: $(k - a)$.
  - Workers supply some of this as savings, getting $(1 + r)$ for each unit of savings
  - Some entrepreneurs might choose optimal k smaller than a, therefore also supplying savings for other entrepreneurs capital demand.

PS3 D: (c) Characterizing SRCE

- If there exists a stationary recursive competitive equilibrium (SRCE), it must be the case that:
  - $\lambda(a, s)$ is stationary.
  - There is a stationary measure of agents choosing $x = 1$, call it μ.

$\Longrightarrow$ w and r are fixed (capital K and labor supply L^S too).

PS3 D: (c) Characterizing SRCE

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How can we argue for a stationary μ?

- Workers choose $x = 1$ if expected value of entrepreneurship exceeds expected value of worker:
$$\mathbb{E}[V(a', s', \epsilon)|s] > \mathbb{E}[W(a', s', \epsilon)|s]$$
- Entrepreneurs choose $x = 1$ under the same criterion.

$\implies$ All agents in state (a, s, ϵ) make the same decision!

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Now recall $\lambda(s, a)$ is stationary

- For each (a, s) there is an ϵ^* s.t. (a, s, ϵ) with $\epsilon > \epsilon^*$ chooses to be an entrepreneur (note that there is an ϵ^* for each (a, s, ϵ)).
- Since $\epsilon \sim U[0, \bar{\epsilon}]$ then the fraction of people (a, s, ϵ) for which $\epsilon > \epsilon^*$ is $\lambda(s, a)[\frac{\bar{\epsilon} - \epsilon^*}{\bar{\epsilon}}]$.

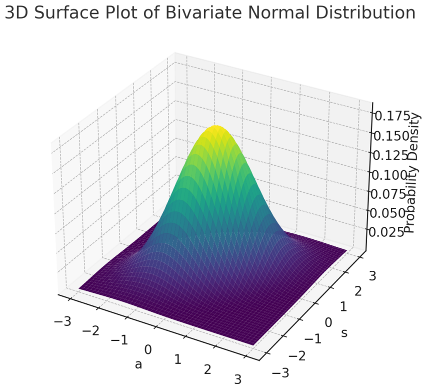
Appendix

How can we imagine a stationary distribution $\lambda(a, s)$?

- Case where (a, s) are continuous
- Case where (a, s) are discrete

How can we imagine a stationary distribution $\lambda(a, s)$?

For illustration, this figure shows the PDF of a bivariate normal[2]. This is a form that our stationary distribution for $\lambda(a, s)$ could take.

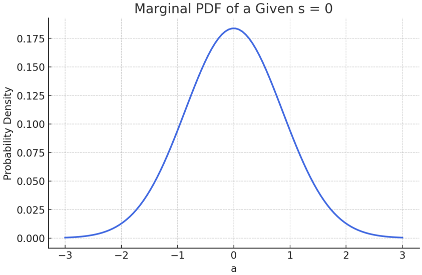


[2] The probability density function (PDF) of the bivariate normal distribution is:

$$f(a, s) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}\begin{bmatrix} a-\mu_a & s-\mu_s \end{bmatrix} \boldsymbol{\Sigma}^{-1} \begin{bmatrix} a-\mu_a \\ s-\mu_s \end{bmatrix}\right)$$

Stationary distribution for wealth (a)

In DS7 and PS3 Problem A, we plot the unconditional distribution of wealth and the conditional on s distribution of wealth. The following picture shows how the conditional on s distribution of wealth is actually a cross-section[3] (2D curve) of the 3D curve in the last slide.
This figure shows how the density changes with a while s remains fixed at 0.

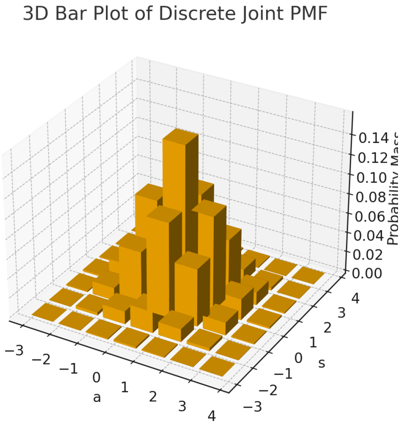


[3] A cross-section at $s = s_0$ means analyzing the function:

$$\lambda(a \mid s = s_0) = \lambda(a, s_0)$$

How can we imagine a stationary distribution $\lambda(a, s)$?

We typically think on the state s as being discrete, and we discretize wealth a to simplify the problem. Then, we actually should think about a Probability Mass Function (PMF), the equivalent of the density but for discrete random variables. For illustration, this figure shows a PMF.



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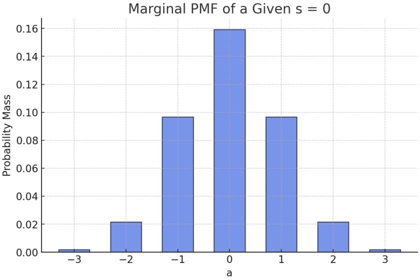

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