

Econ 210B

Discussion Session N°8

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February 28, 2025

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Plan for Today

Aiyagari Diagram

Hugget Model

PS3 Problem D: Aiyagari Model with Occupational Choice (Final 2024)

Stationary Recursive Competitive Equilibrium (SRCE)

Note

- ▶ Asset demand A' becomes capital supply next period $(K^s)'$
- ▶ In the SRCE $K = K'$, $A = A'$

From market clearing:

$$\begin{array}{ll} \text{Capital:} & K^d = K^s \\ \text{Assets:} & A' = (K^s)' \end{array} \Rightarrow K = A$$

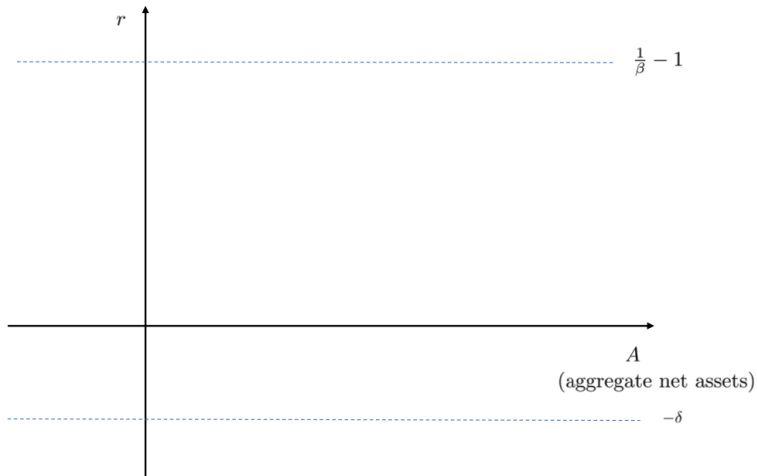
Solution for the SRCE is finding λ^* and r^* s.t.:

$$\lambda^* = \mathcal{H}(\lambda^*) \tag{1}$$

$$K(r^*) = A(r^*; \lambda^*) \tag{2}$$

Aiyagari Diagram

borrowing limit: $a' \geq -\phi$



Practice Question

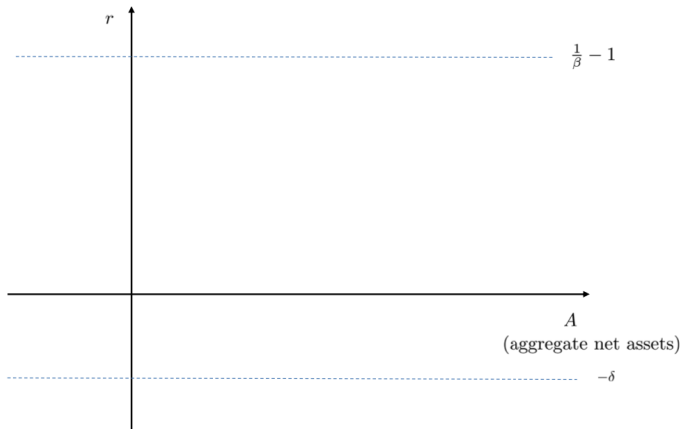
Using the equilibrium of $K(r)$ and $A(r)$

1. Show how equilibrium capital and interest rate would react to a stricter borrowing limit $-\phi' > -\phi$
2. Characterize equilibrium capital and interest rate when $-\phi \rightarrow -\infty$
3. Explain how the result of 2. relates to the complete markets allocation

Practice Question

1. Show how equilibrium capital and interest rate would react to a stricter borrowing limit $-\phi' > -\phi$

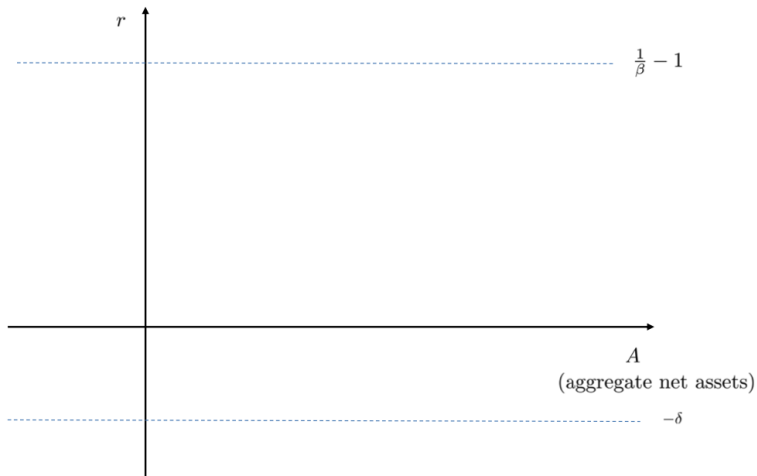
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Practice Question

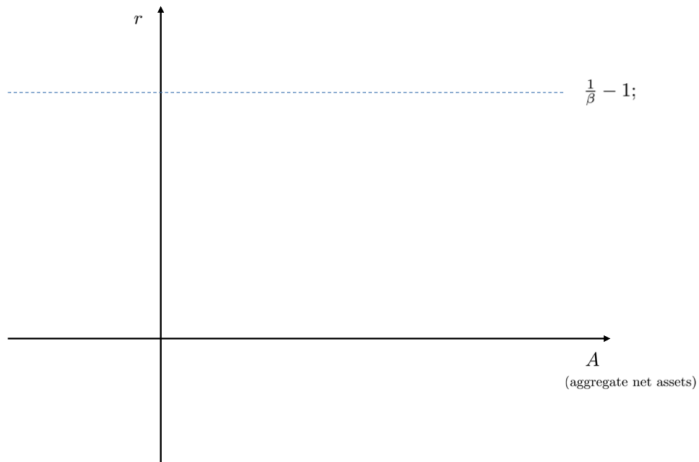
2. Characterize equilibrium capital and interest rate when $-\phi \rightarrow -\infty$

borrowing limit: $a' \geq -\phi$



Aiyagari Model with no Capital (Huggett Model)

borrowing limit: $a' \geq -\phi$



PS3 Problem D: Aiyagari Model with Occupational Choice

Consider a modified version of the Aiyagari model where individuals have an occupation choice.

Individuals face idiosyncratic labor productivity shocks $s \in \{s_L, s_H\}$ and maximize the expected lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t)) \pi(s^t).$$

They earn income either by working at wage w or operating a technology with decreasing returns to scale:

$$\epsilon f(k, n),$$

where k is capital and n is labor.

PS3 D: Occupational Choice and Value Functions

- ▶ Each individual has an entrepreneurial ability ϵ drawn from a uniform distribution on $[0, \bar{\epsilon}]$.
- ▶ The value function of a worker is $W(a, s, \epsilon)$, while that of an entrepreneur is $V(a, s, \epsilon)$.
- ▶ The occupational choice variable $x \in \{0, 1\}$ determines their status:
 - ▶ $x = 0$: Worker
 - ▶ $x = 1$: Entrepreneur

The occupation choice is made one period in advance.

- ▶ The budget constraints are:

$$c + a' = \pi(a, s, \epsilon), \quad (\text{Entrepreneur}) \quad (1)$$

$$c + a' = ws + (1 + r)a. \quad (\text{Worker}) \quad (2)$$

We assume $a' > 0$

PS3 D: Aiyagari Model with Occupational Choice

- (a) Write the maximization problem in recursive form for an individual who is a worker at time t , and for an individual who is an entrepreneur at time t .
- (b) The profits for an entrepreneur can be written as

$$\pi(a, s, \epsilon) = \max_{k, n} \{ \epsilon f(k, n) + (1 - \delta)k - (1 + r)(k - a) - w \max\{n - s, 0\} \}$$

subject to the borrowing constraint:

$$k - a \leq \gamma a, \quad \gamma \in [0, 1].$$

- (i) What factors limit the borrowing capacity of the entrepreneur?
 - (ii) How would you explain the form taken by the cost of factors in the profit function above?
 - (iii) Who owns capital in the problem, and how can you make that determination?
- (c) Go as far as you can in characterizing the solution to the Aiyagari model with occupational choice in a stationary recursive competitive equilibrium.

PS3 D (a): Recursive Maximization Problem

Worker:

$$W(a, s, \epsilon) = \max_{\{c, a', x\}} \{u(c) + \beta ((1-x)\mathbb{E}[W(a', s', \epsilon)] + x\mathbb{E}[V(a', s', \epsilon)])\}$$

$$\begin{aligned} \text{s.t.} \quad & c + a' = ws + (1+r)a, \\ & a' \geq 0, \\ & x \in \{0, 1\}. \end{aligned}$$

Entrepreneur:

$$V(a, s, \epsilon) = \max_{\{c, a', x\}} \{u(c) + \beta (x\mathbb{E}[V(a', s', \epsilon)] + (1-x)\mathbb{E}[W(a', s', \epsilon)])\}$$

$$\begin{aligned} \text{s.t.} \quad & c + a' = \pi(a, s, \epsilon), \\ & a' \geq 0, \\ & x \in \{0, 1\}. \end{aligned}$$

PS3 D: (b) Entrepreneurial Profits

Profits for an entrepreneur are given by:

$$\pi(a, s, \epsilon) = \max_{k, n} \{ \epsilon f(k, n) + (1 - \delta)k - (1 + r)(k - a) - w \max\{n - s, 0\} \}$$
$$\text{s.t. } k - a \leq \gamma a, \quad \gamma \in [0, 1].$$

(i) What factors limit borrowing capacity?

- ▶ $(k - a)$: Excess capital (amount of capital the entrepreneur must borrow in capital/financial markets) is bounded above by the fraction γ of own wealth a .
- ▶ γ : Collateral factor
- ▶ a : Own wealth

PS3 D: (b) Entrepreneurial Profits

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$$\text{s.t. } k - a \leq \gamma a, \quad \gamma \in [0, 1].$$

(ii) Explain costs form

$$c(n, k) = \underbrace{(1 + r)(k - a)}_{\text{Capital cost}} + \underbrace{w \max\{n - s, 0\}}_{\text{Labor cost}}$$

- ▶ For each unit of capital borrowed in the capital market, the entrepreneur must pay $(1 + r)$.
- ▶ If the entrepreneur's choice of labor is higher than her own efficiency labor units (i.e., $n > s$), she must employ the difference $(n - s)$ in labor markets and pay a wage rate w for each unit.

PS3 D: (b) Entrepreneurial Profits

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$$\text{s.t. } k - a \leq \gamma a, \quad \gamma \in [0, 1].$$

(iii) Who owns capital? Why?

$$c(n, k) = \underbrace{(1 + r)(k - a)}_{\text{Capital cost}} + \underbrace{w \max\{n - s, 0\}}_{\text{Labor cost}}$$

- ▶ Some capital is owned by the entrepreneur: a .
- ▶ Some capital is purchased at capital markets: $(k - a)$.
 - ▶ Workers supply some of this as savings, getting $(1 + r)$ for each unit of savings
 - ▶ Some entrepreneurs might choose optimal k smaller than a , therefore also supplying savings for other entrepreneurs capital demand.

PS3 D: (c) Characterizing SRCE

- ▶ If there exists a stationary recursive competitive equilibrium (SRCE), it must be the case that:
 - ▶ $\lambda(a, s)$ is stationary.
 - ▶ There is a stationary measure of agents choosing $x = 1$, call it μ .
- ⇒ w and r are fixed (capital K and labor supply L^S too).

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How can we argue for a stationary μ ?

- ▶ Workers choose $x = 1$ if expected value of entrepreneurship exceeds expected value of worker:

$$\mathbb{E}[V(a', s', \epsilon)|s] > \mathbb{E}[W(a', s', \epsilon)|s]$$

- ▶ Entrepreneurs choose $x = 1$ under the same criterion.

⇒ All agents in state (a, s, ϵ) make the same decision!

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Now recall $\lambda(s, a)$ is stationary

- ▶ For each (a, s) there is an ϵ^* s.t. (a, s, ϵ) with $\epsilon > \epsilon^*$ chooses to be an entrepreneur (note that there is an ϵ^* for each (a, s, ϵ)).
- ▶ Since $\epsilon \sim U[0, \bar{\epsilon}]$ then the fraction of people (a, s, ϵ) for which $\epsilon > \epsilon^*$ is $\lambda(s, a) \left[\frac{\bar{\epsilon} - \epsilon^*}{\bar{\epsilon}} \right]$.

Appendix

How can we imagine a stationary distribution $\lambda(a, s)$?

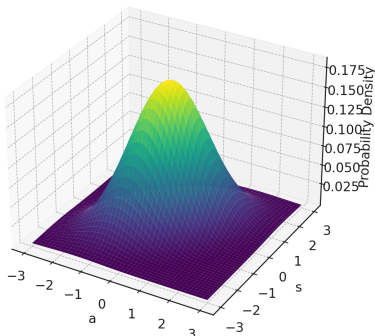
Case where (a, s) are continuous

Case where (a, s) are discrete

How can we imagine a stationary distribution $\lambda(a, s)$?

For illustration, this figure shows the PDF of a bivariate normal². This is a form that our stationary distribution for $\lambda(a, s)$ could take.

3D Surface Plot of Bivariate Normal Distribution



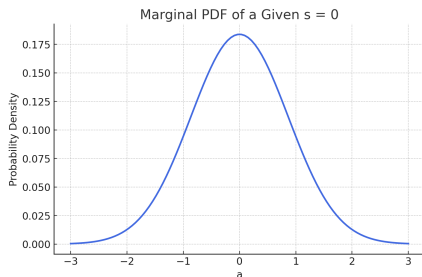
²The probability density function (PDF) of the bivariate normal distribution is:

$$f(a, s) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2} \begin{bmatrix} a - \mu_a & s - \mu_s \end{bmatrix} \Sigma^{-1} \begin{bmatrix} a - \mu_a \\ s - \mu_s \end{bmatrix}\right)$$

Stationary distribution for wealth (a)

In DS7 and PS3 Problem A, we plot the unconditional distribution of wealth and the conditional on s distribution of wealth. The following picture shows how the conditional on s distribution of wealth is actually a cross-section³ (2D curve) of the 3D curve in the last slide.

This figure shows how the density changes with a while s remains fixed at 0.



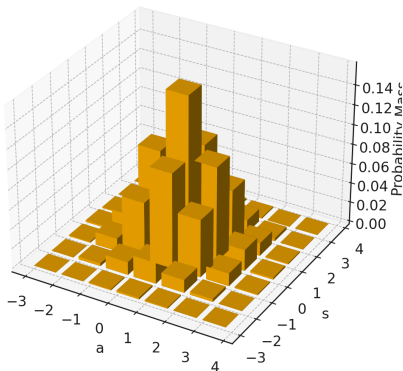
³A cross-section at $s = s_0$ means analyzing the function:

$$\lambda(a \mid s = s_0) = \lambda(a, s_0)$$

How can we imagine a stationary distribution $\lambda(a, s)$?

We typically think on the state s as being discrete, and we discretize wealth a to simplify the problem. Then, we actually should think about a Probability Mass Function (PMF), the equivalent of the density but for discrete random variables. For illustration, this figure shows a PMF.

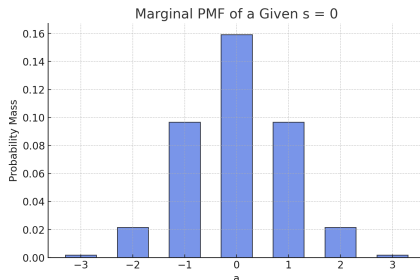
3D Bar Plot of Discrete Joint PMF



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⁴A cross-section at $s = s_0$ means analyzing the function:

$$\lambda(a \mid s = s_0) = \lambda(a, s_0)$$