Econ 210B
Discussion Session N°7

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Plan for Today

Review of Aiyagari Model

Numerical Solution

Practice Question. 210A Final: Neoclassical Model with Employment Risk

Aiyagari Model

1. **Heterogeneous agents**: Each agent experiences an idiosyncratic productivity shock s_t
2. **Incomplete markets**: Agents face a borrowing limit $a_{t+1} \geq -b$

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Today: Review of model for numerical method. For more details check 210A DS9.

Building Block: 1. Heterogeneous Agents

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  - Example: let $s \in S = \{0, 1\}$, then

$$\mathcal{P} = \begin{bmatrix} \pi(0|0) & \pi(1|0) \\ \pi(0|1) & \pi(1|1) \end{bmatrix}$$

Building Block: 2. Incomplete Markets

Each agent/household solves:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t))\pi(s^t)$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t s_t + (1 + r_t)a_t$$

$$a_{t+1} \geq -b$$

$$a_0, s_0 \quad \text{given}$$

where $s_t \in S$ follows a Markov process: $Pr(s_{t+1} = s' \mid s_t = s) \equiv \pi(s' \mid s)$

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Recursive formulation:

$$v(a, s; w, r) = \max_{c, a'} \left\{ u(c) + \beta \sum_{s'} v(a', s'; w', r')\pi(s'|s)] \right\}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq ws + (1 + r)a \\ a' &\geq -b \end{aligned}$$

- state: a, s (aggregate state: r, w)
- choice: c, a'

$\rightarrow$ policy $a' = g(a, s; w, r)$

Building Block: 3. Stationary Distribution

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$$\lambda_{t+1}(a', s') = \sum_{s \in S} \sum_{a : a' = g(a, s)} \lambda_t(a, s) \pi(s' | s)$$

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$$\lambda_{t+1} = \mathcal{H}(\lambda_t)$$

$$\implies \lambda^* = \mathcal{H}(\lambda^*)$$

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- Interpretation of $\lambda(a, s)$:
  1. Agent level: fraction spent in (a, s) in lifetime
  2. Cross-section: fraction of agents in (a, s) at any time

Aggregation and averages

Because of LLN

- Average wealth in the economy

$$\mathbb{E}[a(r)] = \sum_{s \in S} \sum_{a \in A} \lambda(a, s) g(a, s; r)$$

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$$\mathbb{E}[a(r)] = \sum_{s \in S} \sum_{a \in A} \lambda(a, s) g(a, s; r)$$

- Is also the aggregate "saving demand" in the economy to carry into $t + 1$

$$A' = \sum_{s \in S} \sum_{a \in A} \lambda(a, s) g(a, s; r)$$

Firm's Problem

$$\max_{K,N} F(K, N) - (r + \delta)K - wN$$

FONC determines optimality

$$\begin{aligned}
&\text{Labor demand } N^d: & w &= F_N(K, N) \\
&\text{Capital demand } K^d: & r + \delta &= F_K(K, N) \\
& & r &= F_K(K, N) - \delta
\end{aligned}$$

Stationary Recursive Competitive Equilibrium (SRCE)

i. Value fn $v(a,s)$, policy fns $a' = g(a,s), c = c(a,s)$

ii. Firm demands N^d, K^d

iii. Prices r, w

iv. Stationary distribution $\lambda(a,s)$

such that:

1. Given iii., i. solves the HH problem
2. Given iii., ii. solves the firm's problem
3. Markets clear

$$\begin{aligned} &\text{Labor:} \quad N^d = \pi_\infty S^T \\ &\text{Capital:} \quad K^d = K^s \\ &\text{Assets:} \quad A' = (K^s)' \\ &\text{Goods:} \quad \sum_s \sum_a \lambda(a,s)c(a,s) + \delta K = F(L,K) \end{aligned}$$

4. Stationary distribution $\lambda(a,s)$ solves

$$\lambda(a',s') = \sum_{s\in S} \sum_{a:a'=g(a,s)} \lambda(a,s)\pi(s'|s) \quad \forall a', s'$$

Stationary Recursive Competitive Equilibrium (SRCE)

Note

- Asset demand A' becomes capital supply next period $(K^s)'$
- In the SRCE $K = K'$, $A = A'$

  From market clearing:

$$\begin{array}{rl} \text{Capital:} & K^d = K^s \\ \text{Assets:} & A' = (K^s)' \end{array} \quad \Rightarrow \quad K = A$$

Solution for the SRCE is finding λ^* and r^* s.t.:

$$\lambda^* = \mathcal{H}(\lambda^*) \tag{1}$$

$$K(r^*) = A(r^*; \lambda^*) \tag{2}$$

Numerical Solution to the SRCE: Parametrization

- **Utility function:**
$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$
- **Production function:**
$$F(K,N) = AK^{\alpha}N^{1-\alpha}$$
- **Labor income risk:**
  - Each period, the agent is in a state $s_t \in \{0.3, 1\}$.
    - $s_t = 0.3$ (low productivity state).
    - $s_t = 1$ (high productivity state).
  - Labor income in period t is $s_t w$, where w is the equilibrium wage rate.
- **Agent's state evolution:**
  - Follows a Markov chain with transition matrix $\mathcal{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$.
- **Borrowing constraint:**
  - Parameterized by $\phi = 0$.
- **Parameter values:**
  - $A = 1$, $\alpha = 0.33$, $\delta = 0.1$, $\beta = 0.96$, $\gamma = 1.25$.

Numerical Solution to the SRCE: Steps

1. Pick a K, given $N = \pi_\infty S^T$, compute r and w
   - from FONC of firms
2. Solve household's problem under r and w
   - Recall Blackwell + CMT provides an algorithm to find HH solution: VFI
   - Guess V_0 and iterate with the Bellman operator until convergence $TV^* = V^*$
   - Once converged, plug state in V^* and solve FONC for *policy function g*
   - This gives you a' and c
3. Use solution to household problem to compute stationary measure λ^* via $\mathcal{H}$
4. Compute $E[a(r)]$ and determine excess demand for capital

$$K - E[a(r)]$$

   - K being your initial guess for K
5. If excess demand is zero, equilibrium found; if not, adjust K and repeat 1-4.

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Now to matlab!

210A Final: Neoclassical Model with Employment Risk

Consider an economy populated by a continuum of ex-ante identical households. Each household chooses consumption c_t and savings a_{t+1} to maximize expected lifetime utility,

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

subject to a household budget constraint and borrowing limit b,

$$c_t + a_{t+1} \leq w_t s_t + (1 + r_t) a_t$$

$$a_{t+1} \geq -b$$

where s_t represents idiosyncratic employment risk

$$s_t = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p \end{cases}$$

so that $s_t = 1$ represents employment and $s_t = 0$ unemployment. You may assume that wages w_t and interest rates r_t‘ are fixed.

210A Final: Neoclassical Model with Employment Risk

1. Suppose initially that there is no borrowing limit ($b = \infty$) or income uncertainty ($s_t = \bar{s}$). Using the envelope condition, derive the household Euler Equation.
2. Set b to be the "natural borrowing limit". Show how the household Euler Equation changes in this case. Explain.
3. Now consider the full model where –in addition to the borrowing limit– households face stochastic employment risk s_t. Provide an expression for the household Euler Equation in this case.
   Explain how consumption-savings behavior changes relative to the cases in (2) and (1).
4. Is the decentralized equilibrium efficient? Justify your answer.
5. Suppose now that there exists a competitive insurance sector which sells actuarially fair employment insurance. Does this change your answer to part (4)? Explain.

210 Final: (1) No income uncertainty, no borrowing limit

Sequential problem under no borrowing limit or income uncertainty

$$\max_{\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t \bar{s} + (1 + r_t) a_t$$

Assuming wages and interest rates are fixed.

In recursive formulation

$$\begin{aligned} v(a) \quad &= \quad \max_{c, a'} \left\{ u(c) + \beta v(a') \right\} \\ \text{s.t.} \quad & \quad\quad c + a' \leq w\bar{s} + (1 + r)a \end{aligned}$$

210 Final: (1) No income uncertainty, no borrowing limit

Lagrangian:

$$\mathcal{L}(a) = u(c) + \beta v(a') + \theta(w\bar{s} + (1+r)a - c - a')$$

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FONC:

$$\begin{aligned} \{c\}: \quad & c^{-\gamma} && = \theta \\ \{a'\}: \quad & \beta v'(a') && = \theta \end{aligned}$$

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Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \beta v'(a') \tag{1}$$

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Envelope:

$$\frac{\partial v(a)}{\partial a} = \theta(1+r) \quad \rightarrow \quad \frac{\partial v(a')}{\partial a'} = \theta'(1+r')$$

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Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v'(a') = \frac{\partial v(a')}{\partial a'} = c'^{-\gamma}(1+r')$$

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Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v'(a') = \frac{\partial v(a')}{\partial a'} = c'^{-\gamma}(1+r')$$

Replace $v'(a')$ in (1) to arrive to Euler Equation:

$$c^{-\gamma} = \beta c'^{-\gamma}(1+r') \tag{2}$$

210 Final: (2) No income uncertainty, borrowing limit

Set b to be the "natural borrowing limit". Show how the household Euler Equation changes in this case.

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Sequential problem under borrowing limit and no income uncertainty

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\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} & \quad \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \\
\text{s.t.} \quad c_t + a_{t+1} & \leq w_t \bar{s} + (1 + r_t) a_t \\
a_{t+1} & \geq -b
\end{aligned}
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In recursive formulation

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\begin{aligned}
v(a) \quad = \quad & \max_{c, a'} \left\{ u(c) + \beta v(a') \right\} \\
\text{s.t.} \quad & c + a' \leq w\bar{s} + (1+r) \\
& a' \geq -b
\end{aligned}
$$

Wages and the interest rate are assumed constant (makes sense given SRCE).

210 Final: (2) No income uncertainty, borrowing limit

Lagrangian:

$$\mathcal{L}(a) = u(c) + \beta v(a') + \theta(w\bar{s} + (1+r)a - c - a') + \mu(a' + b)$$

$$\begin{aligned} \text{FONC} \quad \{c\}: & \quad c^{-\gamma} && = \theta \\ \{a'\}: & \quad \beta v'(a') + \mu && = \theta \end{aligned}$$

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Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \beta v'(a') + \mu \tag{3}$$

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Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \beta v'(a') + \mu \tag{3}$$

Envelope does not change:

$$\frac{\partial v(a')}{\partial a'} = \theta'(1 + r')$$

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Combining FONC $\{c\}$ and $\{a'\}$:

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Envelope does not change:

$$\frac{\partial v(a')}{\partial a'} = \theta'(1+r')$$

Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v'(a') = \frac{\partial v(a')}{\partial a'} = c'^{-\gamma}(1+r')$$

Replace $v'(a')$ in (3) to arrive to Euler Equation:

$$c^{-\gamma} = \beta c'^{-\gamma}(1+r') + \mu \tag{4}$$

210 Final: (2) No income uncertainty, borrowing limit

Explain how/why EE changes

By complementary slackness condition: $\mu(a' + b) = 0$.

If the constraint does not bind, i.e. $a'(a) > -b$, then $\mu = 0$ and

$$c^{-\gamma} = \beta c'^{-\gamma}(1 + r') \tag{5}$$

Otherwise $(a'(a) \leq b)$, $\mu \geq 0$ and

$$c^{-\gamma} \geq \beta c'^{-\gamma}(1 + r') \tag{6}$$

Meaning the agent consumes less today than optimal (higher marginal utility of consumption today than would hold if EE held with equality), only because their optimal level of borrowing is off-limits.

Not because of precautionary savings.

210 Final: (3) Income uncertainty, borrowing limit

Sequential problem under borrowing limit and income uncertainty

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} & \quad \mathbb{E} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \\ \text{s.t.} \quad c_t + a_{t+1} & \leq w_t s_t + (1+r_t) a_t \\ a_{t+1} & \geq -b \end{aligned}$$

In recursive formulation

$$\begin{aligned} v(a,s) &= \max_{c,a'} \{u(c) + \beta \mathbb{E} v(a',s')\} \\ \text{s.t.} & \quad c + a' \leq ws + (1+r) \\ & \quad a' \geq -b \end{aligned}$$

Wages and the interest rate are assumed constant (makes sense given SRCE).

210 Final: (3) Income uncertainty, borrowing limit

Note

$$\mathbb{E}v(a',s') = pv(a',1) + (1-p)v(a',0)$$

Where $v(a,1)$:

$$\begin{aligned} v(a,1) &= \max_{c,a'} \{u(c) + \beta \mathbb{E}v(a',s')\} \\ \text{s.t.} \quad & c + a' \leq w + (1+r) \\ & a' \geq -b \end{aligned}$$

And $v(a,0)$:

$$\begin{aligned} v(a,0) &= \max_{c,a'} \{u(c) + \beta \mathbb{E}v(a',s')\} \\ \text{s.t.} \quad & c + a' \leq (1+r) \\ & a' \geq -b \end{aligned}$$

210 Final: (3) Income uncertainty, borrowing limit

Lagrangian:

$$\mathcal{L}(a,s) = u(c) + \beta\mathbb{E}v(a',s') + \theta(ws + (1+r)a - c - a') + \mu(a' + b)$$

$$\begin{aligned} \text{FONC} \quad \{c\}: &\quad c^{-\gamma} &= \theta \\ \{a'\}: &\quad \beta\mathbb{E}v_a(a',s') + \mu &= \theta \end{aligned}$$

Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \mathbb{E}v_a(a',s') + \mu \tag{7}$$

Envelope:

$$\frac{\partial v(a',s')}{\partial a'} = \theta'(1+r')$$

Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v_a(a',s') = \frac{\partial v(a',s')}{\partial a'} = c'^{-\gamma}(1+r')$$

Replace $v_a(a',s')$ in (7) to arrive to Euler Equation:

$$c^{-\gamma} = \beta\mathbb{E}[c'^{-\gamma}(1+r')] + \mu \tag{8}$$

210 Final: (3) Income uncertainty, borrowing limit

Explain how/why EE changes

By complementary slackness condition: $\mu(a' + b) = 0$.
If the constraint does not bind, i.e. $a'(a, s) > -b$, then $\mu = 0$ and

$$c^{-\gamma} = \beta \mathbb{E}[c'^{-\gamma}(1 + r')] \tag{9}$$

Note that this does not mean the household choice for consumption and savings is the same as (1) and (2).

Even if $\bar{s} = \mathbb{E}(s) = p$ (so the type in the previous questions is the expected productivity), $u'(\mathbb{E}(s') + (1 + r)a') = u'(\bar{s} + (1 + r)a') < \mathbb{E}(u'(s' + (1 + r)a'))$ because the utility function exhibits Prudence/convex marginal utility (i.e. $u''' > 0$).

Precautionary savings appear to avoid the chances of an event with very low utility in the future, so the household will save more or borrow less even if $\mu = 0$ -constraint doesn't bind-.

210 Final: (3) Income uncertainty, borrowing limit

... Explain how/why EE changes

Otherwise ($a'(a,s)^* \leq b$), $\mu \geq 0$ and

$$c^{-\gamma} \geq \beta \mathbb{E} c'^{-\gamma}(1+r') \tag{10}$$

Meaning the constrained agent of course consumes less today than optimal (higher marginal utility of consumption today that would hold if the equation held with equality), because their optimal level of borrowing is off-limits and also because of precautionary savings.

Again, the result is higher savings (less borrowing) than (1) and (2).

210A Final: Efficiency under borrowing limit and uncertainty

4. Is the decentralized equilibrium efficient?

   No. There will be higher savings (and therefore higher capital) than in the complete markets equilibrium. The interest rate will also be lower than the efficient one.

5. Does fair insurance change your answer to (4)?

   Yes. The existence of actuarially fair insurance completes the market and therefore this formulation of competitive equilibrium has the efficient capital market allocation.