Plan for Today

Midterm Review

Problem A: Environmental Quality and Flow Policies

Context: The problem considers a social planner maximizing an objective function that depends on consumption and environmental quality. The evolution of environmental quality follows a dynamic equation, and the planner chooses consumption over time.

Objective Function:

$$\max_{c(t)} \int_0^\infty e^{-\rho t} \left[\sigma \ln c(t) + (1-\sigma) \ln S(t)\right] dt$$

where $\sigma \in (0, 1)$, $\rho > 0$, and

$$\dot{S}(t) = \left(1 - \frac{S(t)}{2}\right) S(t) - c(t), \quad c(t) \geq 0, \quad S(t) \geq 0.$$

Problem A: (a) Hamiltonian Formulation

Question: Write the *current value* Hamiltonian associated with the optimization problem for the planner, and indicate the state variable and the control variable(s). Define the co-state variable $\mu(t)$.

$$\tilde{\mathcal{H}} = \sigma \ln c(t) + (1-\sigma) \ln S(t) + \mu(t)\left[\left(1 - \frac{S(t)}{2}\right) S(t) - c(t)\right]$$

control: $c(t)$
state: $S(t)$
co-state: $\mu(t)$

Problem A: (b) Optimality Conditions

Control: $c(t)$
state: $S(t)$
co-state: $\mu(t)$

Question: Obtain the optimality conditions for the control, the co-state, and the state variables. Explain how you can determine the transversality condition. Provide an economic interpretation of the co-state variable.

$$\max_{c(t)} \int_0^\infty e^{-\rho t} \overbrace{[\sigma \ln c(t) + (1-\sigma)\ln S(t)]}^{g(c(t),\, S(t))} \, dt$$

$$\dot{S}(t) = \left(1 - \frac{S(t)}{2}\right) S(t) - c(t), \quad c(t) \geq 0, \quad S(t) \geq 0.$$

$$\tilde{\mathcal{H}} = \sigma \ln c(t) + (1-\sigma)\ln S(t) + \mu(t)\left[\left(1 - \frac{S(t)}{2}\right) S(t) - c(t)\right]$$

(c) $\quad \dfrac{\sigma}{c(t)} = \mu(t)$

(S) $\quad \dot{\mu}(t) - \rho\mu(t) = -\tilde{H}_S \iff \dot{\mu}(t) - \rho\mu(t) = -\left[\dfrac{1-\sigma}{S(t)} + \mu(t) - \mu(t)S(t)\right]$

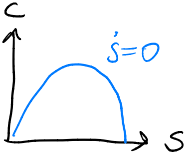
(μ) $\quad \dot{S}(t) = \left(1 - \dfrac{S(t)}{2}\right) S(t) - c(t)$

(TVC) $\quad \lim_{t \to \infty} e^{-\rho t} \mu(t) S(t) = 0 \quad ; \quad \lim_{t \to \infty} e^{-\rho t} \mu(t) \geq 0$

Problem A: (c) Steady State Analysis

① $\dot{S} = 0 \quad S(t) = S^{ss}$ ★

Question: Define the steady state $S^* > 0$ such that $\boxed{\dot{S} = 0}$. Characterize the steady state. Does it always exist? Is it unique? Can the steady state be $S^* > 2$? Explain and show your work.



(M) $\dot{S}(t) = \left(1 - \frac{S(t)}{2}\right) S(t) - c(t)$

$0 = \left(1 - \frac{S}{2}\right) S - c(t)$

$c(t) = \left(1 - \frac{S}{2}\right) S$ ★ $S^* < 2$

② $c(t) = c^* \quad c^{ss} \quad c$

$\boxed{\frac{\sigma}{\mu} = \left(1 - \frac{S}{2}\right) S}$ (1)

$\dot{S} = 0 \Rightarrow \dot{c} = 0 \Rightarrow \dot{\mu} = 0$

③ $\frac{\sigma}{c} = \mu(t)$

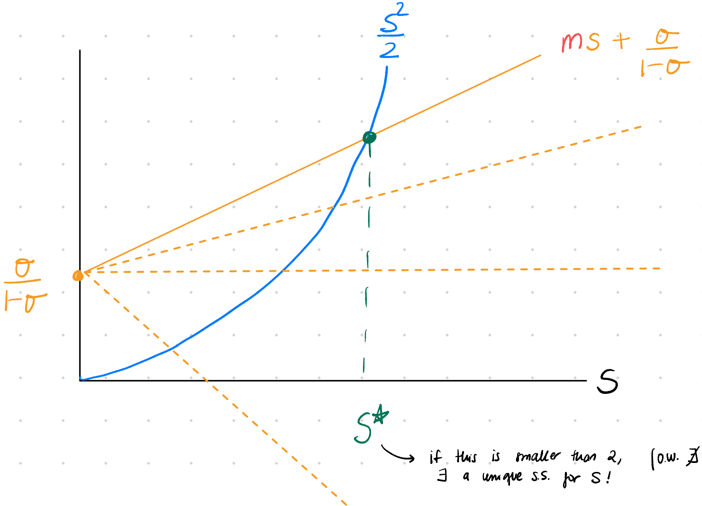
(S) $\dot{\mu}(t) - \rho\mu(t) = -\left[\frac{1-\sigma}{S(t)} + \mu(t) - \mu(t)S(t)\right]$

(2)

$0 - \rho\mu = -\left[\frac{1-\sigma}{S} + \mu - \mu S\right] \Rightarrow \boxed{\rho\mu = \frac{1-\sigma}{S} + \mu - \mu S}$

Putting (1) and (2) together you'll reach an equation of the form:

$$\underbrace{\left(1 - \frac{\sigma}{1-\sigma}(1-\rho)\right)}_{m} S + \frac{\sigma}{1-\sigma} = \frac{S^2}{2}$$



if this is smaller than 2, $\exists$ a unique s.s. for S! (o.w. $\nexists$ ss for S)

Problem A: (d)-(e) Decentralized Version

$g(x(t), y(t), t)$
$= i(t) \cdot \tilde{g}(x(t), y(t))$

Setup: The decentralized version corresponds to a representative household maximizing the same objective function but with $Z(t)$ instead of $S(t)$.

Objective Function:

$= \underbrace{e^{-\rho t}} \cdot \underbrace{\tilde{g}(x(t), y(t))}$

$$\max_{c(t), x(t)} \int_0^\infty e^{-\rho t} \left[\sigma \ln c(t) + (1 - \sigma) \ln Z(t)\right] dt$$

The system evolves as:

$$\dot{S}(t) = \left(1 - \frac{S(t)}{2}\right) S(t) - x(t), \quad c(t) = x(t)(1 - \tau(t)) + T(t).$$

Question: Write the *present value* Hamiltonian and derive the optimality conditions. Define the co-state variable $\theta(t)$.

$$\max_{c(t),x(t)} \int_0^\infty e^{-\rho t}\left[\sigma \ln c(t) + (1-\sigma)\ln Z(t)\right] dt$$

$\theta(t)$: co-state

$x(t)$: control

$S(t)$: state

The system evolves as:

$$\dot{S}(t) = \left(1 - \frac{S(t)}{2}\right) S(t) - x(t), \quad c(t) = x(t)(1-\tau(t)) + T(t).$$

$$\mathcal{H} = e^{-\rho t}\left[\sigma \ln\left[\underbrace{x(t)(1-\tau(t)) + T(t)}_{c(t)}\right] + (1-\sigma)\ln Z(t)\right]$$

$$+ \ \theta(t)\left[\left(S(t) - \frac{S(t)^2}{2} - x(t)\right)\right]$$

(x) $\quad e^{-\rho t}\dfrac{\sigma}{c(t)}(1-\tau(t)) = \theta(t)$

(S) $\quad \dot{\theta}(t) = -\left[\theta(t) - \theta(t)S(t)\right]$

(θ) $\quad \dot{S}(t) = \left(1 - \dfrac{S(t)}{2}\right)S(t) - x(t)$

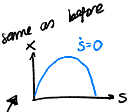
(TVC) $\quad$ '' $\quad$ ''

Problem A: (f) Planner vs Decentralized Outcomes $\tau(t) = T(t) = 0$

Question: Compare the optimality conditions of the decentralized version and the social planner. How do they differ? In which case is S in steady state higher or lower? Explain.

Assume $\tau(t) = T(t) = 0$

(x) becomes $e^{-\rho t} \dfrac{\sigma}{X(t)} = \theta(t)$



Now we will try to solve for the S.S. for S.

(1) $\dot{S} = 0$: (θ) $0 = \left(1 - \frac{S}{2}\right) S - X(t) \Rightarrow X(t) = \left(1 - \frac{S}{2}\right) S$

so we learn that if S is in SS, X(t) does not depend on t, hence its on SS.

Does $\dot{x} = 0$ buy us $\dot{\theta} = 0$? No :(

(x) $e^{-\rho t} \dfrac{\sigma}{x} = \theta(t)$: still there's a t ☹

We'll follow a different route ☺

(2) from (x) [without imposing $\dot{c} = 0$]: $\dot{\theta}(t) = \dfrac{d\theta(t)}{dt} = -\rho e^{-\rho t} \dfrac{\sigma}{X(t)} - \dfrac{\sigma}{X(t)^2} \dot{X}(t) e^{-\rho t}$

Problem A: (f) Planner vs Decentralized Outcomes

Question: Compare the optimality conditions of the decentralized version and the social planner. How do they differ? In which case is S in steady state higher or lower? Explain.

... 2 $\quad \dot{\theta}(t) = -e^{-\rho t} \frac{\sigma}{x(t)} \left[\rho + \frac{\dot{x}(t)}{x(t)} \right]$ / Recall $\theta(t) = e^{-\rho t} \frac{\sigma}{x(t)}$

$\div\ \theta(t)$ both sides

$$\frac{\dot{\theta}(t)}{\theta(t)} = -\left[\rho + \frac{\dot{x}(t)}{x(t)} \right]$$

Now bring (5) on : $\dot{\theta}(t) = -(\theta(t) - \theta(t) s(t))$ / $\div\ \theta(t)$

$$\frac{\dot{\theta}(t)}{\theta(t)} = -(1 - s(t))$$

So for any t (in ss or not), the following holds :

$$\rho + \frac{\dot{x}(t)}{x(t)} = -(1 - s(t))$$

Lastly, impose $\dot{s} = 0 \Rightarrow s(t) = s^{ss}$; $\dot{x}(t) = 0$

$$\rho = -(1 - s^{ss})$$

$$s^{ss} = \rho + 1 \quad //$$

Problem A: (f) Planner vs Decentralized Outcomes

Question: Compare the optimality conditions of the decentralized version and the social planner. How do they differ? In which case is S in steady state higher or lower? Explain.

$$S^D = \rho + 1$$

$$S^P = \text{solution to} \quad \left(1 - \frac{\sigma}{1-\sigma}(1-\rho)\right)S^P + \frac{\sigma}{1-\sigma} = \frac{(S^P)^2}{2}$$

I would expect $S^D < S^P$ because consumer does not internalize the fact that they consumption affects $Z(t)$, then they "over consume."

But this difference will depend on σ and ρ. For instance, the difference fades as $\sigma \to 1$, meaning we value less $S(t)/Z(t)$, so the externality is less relevant.

Numerical Example :

Take $\rho = 0$ [very patient consumer], $\sigma = 0.5$ (same value for $C(t)$ and $S(t)/Z(t)$)

Then $\left.\begin{array}{l} S^P = \sqrt{2} \\ S^D = 1 \end{array}\right\} S^P > S^D$!

Problem A: (g) Policy Intervention in Decentralization

Question: Can a tax (or subsidy) $\tau(t)$ align the decentralized solution with the social planner's allocation? Why or why not?
Use the condition $T(t) = \tau(t)x(t)$ to ensure budget balance.

$$e^{-\rho t} \frac{\sigma}{c(t)} (1 - \tau(t)) = \theta(t)$$

: Tax DOES desincentivize consumption.

But is it enough to correct the distortion? NO.

It is sufficient to show that the tax does not work to implement the steady state S^P, and if $\dot{\tau} = 0$, it doesn't impact the ss.

Key Take-away : Tax to flow cannot correct the externality ; we need a law of motion for $\tau(t)$!

Problem B: Lucas Trees in Finite Time

Context: The problem considers an economy with two agents, $i = \{A, B\}$, who live for two periods ($t = 1, 2$).
The economy has two trees that produce stochastic output, and agents trade assets competitively in markets.
Both agents maximize expected present value of lifetime utility with logarithmic preferences and discount factor $\beta \in (0, 1)$.

Stochastic Output:

$$\text{Tree 1: } y^1(s) = 1 + s, \quad \text{Tree 2: } y^2(s) = 2(1 - s),$$

where s takes values 0 or 1 with equal probability.

Notation:

- Agents trade assets at market prices P_t^1 and P_t^2. P_1^1 P_1^2
- Denote the holdings of Tree 1 by agent i at the beginning of time t as $z_{1,t}^i$, and for Tree 2 by $z_{2,t}^i$.
- The total existing stocks for each tree are normalized to 1.

Problem B: (a) Budget Constraint

Question: Write the $t = 1$ and $t = 2$ budget constraints for agent i.

SOURCES USES

$t=1$ $$z^i_{1,1}(P^1_1 + 1) + z^i_{2,1}(P^2_1 + 1) = c^i_1 + z^i_{1,2} \cdot P^1_1 + z^i_{2,2} P^2_1$$

$t=2$ $$z^i_{1,2}\, y^1(s) + z^i_{2,2}\, y^2(s) = c^i_2(s)$$

$s = \{0, 1\}$

$i = \{A, B\}$

Two budgets constraints in $t=2$, one for $s=0$ and one for $s=1$

if $s=0$: $z^i_{1,2} + z^i_{2,2} \cdot 2 = c^i_2(0)$

$s=1$: $z^i_{1,2} \cdot 2 + 0 = c^i_2(1)$

Tree 1: $y^1(s) = 1 + s$, Tree 2: $y^2(s) = 2(1 - s)$,

Problem B: (b) Lagrangian and Optimality Conditions

Question: Write the Lagrangian for agent i and obtain the corresponding optimality conditions.

Objective function :
$$\max_{\{c_1^i,\, c_2^i(0),\, c_2^i(1)\}} \left\{ \ln c_1^i + \beta \sum_s \pi(s) \ln c_2^i(s) \right\}$$

Budget constraints :
$$\text{s.t.} \quad z_{1,1}^i (P_1^1 + 1) + z_{2,1}^i (P_1^2 + 1) = c_1^i + z_{1,2}^i \cdot P_1^1 + z_{2,2}^i P_1^2 \qquad (t=1)$$
$$z_{1,2}^i\, y^1(s) + z_{2,2}^i\, y^2(s) = c_2^i(s) \qquad (\text{for } s=\{0,1\},\ t=2)$$

$$\mathcal{L}: \quad \ln c_1^i + \theta_1^i \left(z_{1,1}^i (P_1^1 + 1) + z_{2,1}^i (P_1^2 + 1) - c_1^i - z_{1,2}^i P_1^1 - z_{2,2}^i P_1^2 \right)$$
$$+ \beta \sum_s \left\{ \pi(s) \ln c_2^i(s) + \theta_2^i(s) \left[z_{1,2}^i\, y^1(s) + z_{2,2}^i\, y^2(s) - c_2^i(s) \right] \right\}$$

Problem B: (b) Lagrangian and Optimality Conditions

Question: Write the Lagrangian for agent i and obtain the corresponding optimality conditions.

$$\mathcal{L}: \quad \ln c_1^i + \theta_1^i \left(z_{1,1}^i (p_1^1 + 1) + z_{2,1}^i (p_1^2 + 1) - c_1^i - z_{1,2}^i p_1^1 - z_{2,2}^i p_1^2 \right)$$
$$+ \beta \sum_s \left\{ \pi(s) \ln c_2^i(s) + \theta_2^i(s) \left[z_{1,2}^i \; y^1(s) + z_{2,2}^i \; y^2(s) - c_2^i(s) \right] \right\}$$

$\{c_1^i\}$ $\qquad \dfrac{1}{c_1^i} = \theta_1^i$

$\{c_2^i(s)\}$ $\qquad \cancel{\beta} \dfrac{\pi(s)}{c_2^i(s)} = \cancel{\beta}\, \theta_2^i(s) \rightarrow \theta_2^i(s) = \dfrac{\pi(s)}{c_2^i(s)}$

$\{z_{1,2}^i\}$ $\qquad -\theta_1^i p_1^1 + \beta \sum_s \theta_2^i(s)\, y^1(s) = 0 \rightarrow p_1^1 = \beta \sum_s \dfrac{\theta_2^i(s)}{\theta_1^i} \; y^1(s)$

$\{z_{2,2}^i\}$ $\qquad -\theta_1^i p_1^2 + \beta \sum_s \theta_2^i(s)\, y^2(s) = 0 \rightarrow p_1^2 = \beta \sum_s \dfrac{\theta_2^i(s)}{\theta_1^i} \; y^2(s)$

Problem B: (b) Lagrangian and Optimality Conditions

Question: Write the Lagrangian for agent i and obtain the corresponding optimality conditions.

(The same but with the summations open) Note that the multipliers are not the same just bcs I did not put them inside the β by choice.

$$\max_{\{c_1^i,\, c_2^i(0),\, c_2^i(1)\}} \left\{ \ln c_1^i + \beta \cdot 0.5 \cdot \ln c_2^i(0) + \beta \cdot 0.5 \cdot \ln c_2^i(1) \right\}$$

s.t.

(1) $z_{1,1}^i (P_1^1 + 1) + z_{2,1}^i (P_1^2 + 1) = c_1^i + z_{1,2}^i P_1^1 + z_{2,2}^i P_1^2 \qquad (t=1)$

(2) $z_{1,2}^i + 2 z_{2,2}^i = c_2^i(0) \qquad (t=2, s=0)$

(3) $2 z_{1,2}^i = c_2^i(1) \qquad (t=2, s=1)$

$$\mathcal{L} = \ln c_1^i + \beta \cdot 0.5 \cdot \ln c_2^i(0) + \beta \cdot 0.5 \cdot \ln c_2^i(1) + \gamma_1^i \left[z_{1,1}^i (P_1^1 + 1) + z_{2,1}^i (P_1^2 + 1) - c_1^i - z_{1,2}^i P_1^1 - z_{2,2}^i P_1^2 \right]$$
$$+ \gamma_2^i(0) \left[z_{1,2}^i + 2 z_{2,2}^i - c_2^i(0) \right] + \gamma_2^i(1) \left[2 z_{1,2}^i - c_2^i(1) \right]$$

$\{c_1^i\}$, $\{c_2^i(0)\}$, $\{c_2^i(1)\}$, $\{z_{1,2}^i\}$, $\{z_{2,2}^i\}$

$$\mathcal{L} = \ln c_1^i + \beta \cdot 0.5 \cdot \ln c_2^i(0) + \beta \cdot 0.5 \cdot \ln c_2^i(1) + \gamma_1^i \left[z_{1,1}^i (p_1^1 + 1) + z_{2,1}^i (p_1^2 + 1) - c_1^i - z_{1,2}^i p_1^1 - z_{2,2}^i p_1^2 \right]$$

$$+ \gamma_2^i(0) \left[z_{1,2}^i + 2 z_{2,2}^i - c_2^i(0) \right] + \gamma_2^i(1) \left[2 z_{1,2}^i - c_2^i(1) \right]$$

$\{c_1^i\}$ $$\frac{1}{c_1^i} = \gamma_1^i$$

$\{c_2^i(0)\}$ $$\beta\, 0.5\, \frac{1}{c_2^i(0)} = \gamma_2^i(0)$$

$\{c_2^i(1)\}$ $$\beta\, 0.5\, \frac{1}{c_2^i(1)} = \gamma_2^i(1)$$

$\{z_{1,2}^i\}$ $$\gamma_1^i\, p_1^1 = \gamma_2^i(0) + \gamma_2^i(1) \cdot 2 \rightarrow p_1^1 = \frac{\gamma_2^i(0)}{\gamma_1^i} + 2\, \frac{\gamma_2^i(1)}{\gamma_1^i}$$

$\{z_{2,2}^i\}$ $$\gamma_1^i\, p_1^2 = \gamma_2^i(0) \cdot 2 \rightarrow p_1^2 = 2\, \frac{\gamma_2^i(0)}{\gamma_1^i}$$

From $\{z_{2,2}^i\}$ $\quad p_1^2 = 2\,\dfrac{\gamma_2^i(0)}{\gamma_1^i} \quad \Rightarrow \quad \dfrac{\cancel{2}\,\gamma_2^A(0)}{\gamma_1^A} = \dfrac{\cancel{2}\,\gamma_2^B(0)}{\gamma_1^B}$

$$\frac{\gamma_2^A(0)}{\gamma_1^A} = \frac{\gamma_2^B(0)}{\gamma_1^B} \qquad (*)$$

Note $\{c_1^i\}$ $\quad \dfrac{1}{c_1^i} = \gamma_1^i$

$\{c_2^i(0)\}$ $\quad \beta\, 0.5\, \dfrac{1}{c_2^i(0)} = \gamma_2^i(0)$

In $(*)$ $\quad \cancel{\beta}\ \cancel{0.5}\ \dfrac{1}{c_2^A(0)}\ c_1^A = \cancel{\beta}\ \cancel{0.5}\ \dfrac{1}{c_2^B(0)}\ c_1^B$

$$\boxed{\frac{c_1^A}{c_2^A(0)} = \frac{c_1^B}{c_2^B(0)}} \equiv C \quad \text{(a constant)}$$

$\{z_{1,2}^i\}$ $\quad p_1^1 = C + 2\,\dfrac{\gamma_2^i(1)}{\gamma_1^i} \Rightarrow \dfrac{\gamma_2^A(1)}{\gamma_1^A} = \dfrac{\gamma_2^B(1)}{\gamma_2^B} \Rightarrow \boxed{\dfrac{c_1^A}{c_2^A(1)} = \dfrac{c_2^B}{c_2^B(1)}}$

: Risk sharing!

Problem B: (c) Competitive Equilibrium Conditions

Question: State the conditions for a competitive equilibrium in this economy, ensuring all market clearing conditions are included.

A competitive equilibrium will be: (Given initial holding for assets $z^i_{1,1}, z^i_{2,1}$)

(A) allocations: $c^i_1, z^i_{1,2}, z^i_{2,2}$ for $i = \{A, B\}$

$c^i_2(s)$ for $i = \{A, B\}$ $s = \{0, 1\}$

(B) Prices: P^1_1, P^2_2

such that

(1) Agents A and B solve the probeem in (b) [max utility given B.C.]

(2) Markets clear:

Tree 1 $z^A_{1,t} + z^B_{1,t} = 1$, $t = \{1, 2\}$

Tree 2 $z^A_{2,t} + z^B_{2,t} = 1$, $t = \{1, 2\}$

Output / Fruit Goods mkt:

$t=1$ $c^A_1 + c^B_1 = 2$

$t=2$ $c^A_2(s) + c^B_2(s) = y^1(s) + y^2(s)$
$= 3 - s$, $s = \{0, 1\}$

Problem B: (d) Initial Stock Allocation and Market Prices

Setup: Suppose initial stock holdings are:

$$z_{1,1}^A = 1, \quad z_{2,1}^A = 0, \quad z_{1,1}^B = 0, \quad z_{2,1}^B = 1.$$

Question: Characterize the stock prices and consumption allocations. Are agents able to fully insure? Explain.

Are agents able to fully insure? yes. markets are complete

From F.O.C :

$$\frac{c_2^A(0)}{c_1^A} = \frac{c_2^B(0)}{c_1^B} \quad \text{and} \quad \frac{c_2^A(1)}{c_1^A} = \frac{c_2^B(1)}{c_2^B}$$

(see pdf slide n°20)

For each state of the world, consumption growth is equalized across agents.

I used

(1) $$\frac{C_2^A(0)}{C_1^A} = \frac{C_2^B(0)}{C_1^B}$$

(2) $$\frac{C_2^A(1)}{C_1^A} = \frac{C_2^B(1)}{C_2^B}$$

(3) $$p_1^2 = 2\frac{\gamma_2^i(0)}{\gamma_1^i} = \beta\frac{C_1^i}{C_2^i(0)} \quad \text{for } A, B$$

(4) $$p_1^1 = \frac{\gamma_2^i(0)}{\gamma_1^i} + 2\frac{\gamma_2^i(1)}{\gamma_1^i} = 0.5\beta\frac{C_1^i}{C_2^i(0)} + \beta\frac{C_1^i}{C_2^i(1)} \quad \text{for } A, B$$

(5) $$C_1^A = 2 - C_1^B$$

(6) $$C_2^A(0) = 3 - C_2^B(0)$$

(7) $$C_2^A(1) = 2 - C_2^B(1)$$

and find

$$\frac{C_1^B}{C_2^B(0)} = \frac{C_1^A}{C_2^A(0)} = \frac{2}{3}$$

$$\frac{C_1^B}{C_2^B(1)} = \frac{C_1^A}{C_2^A(1)} = 1$$

$$P_1^2 = \frac{2}{3}\beta$$

$$P_1^1 = \frac{4}{3}\beta$$

Problem B: (e) Alternative Initial Stock Allocation

Setup: Suppose initial stock holdings are:

$$z^A_{1,1} = z^A_{2,1} = z^B_{1,1} = z^B_{2,1} = 0.5.$$

Question: Will stock prices differ compared to part (d)? Why or why not? Explain and show your work.

Stock prices will not differ.