# Econ 210B
# Discussion Session N°5

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# Plan for Today

Review: Date 0 vs Sequential Markets

Lucas Tree From Class

PS2 Problem E: Risk Sharing

PS2 Problem D: Lucas Tree and Storage Technology (Midterm 2024)

# Goals for Today

- We want to be sure we can set up a general Lucas Tree problem
- For that we need
  1. Some understanding of what Arrow Debreu Date 0 and Arrow Debreu Sequential Markets are
  2. A lot of practice!
- We will do a quick review and then jump to 3 practice questions:
  1. Class Example of Lucas Tree Economy
  2. Problem Set 2, Problem E (Part II): Risk Sharing and Lucas Tree Economy
  3. Problem Set 2, Problem D: Lucas Tree and Storage Technology

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    2. Recursive Competitive Equilibrium (RCE) (Lecture 6, DS4)
    3. Arrow Debreu Date-0 Equilibrium (Lecture 6)

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  3. Arrow Debreu Date-0:
     - All trade happens at date 0. There is a price for consumption in each period(-state). When a period(-state) is realized, agents settle their obligations by exchanging these claims for consumption, but no further security trading takes place.
     - There is only one budget constraint for the household, corresponding to $t = 0$

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     - There is only one budget constraint for the household, corresponding to $t = 0$
- Sounds familiar?

# Date 0 vs Sequential Markets: Arrow Debreu Securities

## Arrow Date-0 Economy

Equilibrium concept: An equilibrium is allocations $\{c_t^i(s^t)\}_{i=1}^N$ and prices $q_t^0(s^t)$ for which:

1. Households Optimize

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t))\pi(s^t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} c_t^i(s^t) q_t^0(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} y_i^t(s^t) q_t^0(s^t)$$

2. Markets clear

$$\sum_{i=1}^N y_i^t(s^t) = \sum_{i=1}^N c_t^i(s^t) \quad \forall t, s^t$$

## Arrow Sequential Markets

Equilibrium concept: An equilibrium is allocations $\{c_t^i(s^t)\}_{i=1}^N$ and prices $Q_t(s_{t+1}|s^t)$ for which:

1. Households Optimize

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t))\pi(s^t)$$

$$\text{s.t.} \quad c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1} \mid s^t) \leq y_i^t(s^t) + a_i^t(s^t), \quad \forall t, s^t$$

2. Markets clear

$$\sum_{i=1}^N y_i^t(s^t) = \sum_{i=1}^N c_t^i(s^t) \quad \forall t, s^t$$

$$\sum_{i=1}^N a_{t+1}^i(s_{t+1}, s^t) = 0 \quad \forall t, s^t$$

# Asset Pricing: Notation

$s_t$: event happening at t

$s^t$: state, history of events that happened until t $\{s_0, s_1, ..., s_t\}$

$\pi(s^t)$: probability of state $s^t$

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- $Q_t(s_{t+1} \mid s^t)$: price at $t$ $(s^t)$ of an AS security that pays 1 at $t+1$ if $s_{t+1}$ realizes

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- $a_{t+1}(s_{t+1}, s^t)$: holdings chosen at $t$ ($s^t$), to carry into $t+1$, of AS security that pays 1 at $t+1$ if $s_{t+1}$ realizes

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$\rightarrow$ Equivalence: $q_{t+1}^0(s^{t+1}) = Q_t(s_{t+1} \mid s^t)\ q_t^0(s^t)$ (under Complete Markets!)

# Lucas Tree From Class

- Large number $N$ of infinitely lived agents starting at $t = 0$, each owning a tree that bears $d(s_t)$ fruit (output) each period, process for $d(s_t)$ has Markov property: $\pi(s_t|s^{t-1}) = \pi(s_t|s_{t-1})$.
- Fruit cannot be stored, has to be consumed or sold.
- Each period there is a market for trees (or stocks on trees), with ex-dividend price $P(s^t)$, stock holdings at the beginning of time $t$ are denoted by $z_t$.
- Each period there is a market for risk-free one-period bond paying a gross return $R_t$, bond holdings at the beginning of time $t$ are denoted by $b_t$.
- Each period there are complete markets for Arrow Sequential (AS) securities, AS holdings at the beginning of time $t$, in state $s^t$ are denoted by $a_t(s^t)$.
- There are "natural" borrowing constraints for bonds and AS, and short-sale constraints for trees.

# Lucas Tree From Class

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t))\pi(s^t)$$

- $N$ identical agents, perfectly competitive markets.
- **Assets**: Risk-free bonds, Stocks (claims on fruit), AS securities.

Source of funds:

$$z_t^i\big(d(s_t) + P_t(s^t)\big) + a_t^i(s^t) + R_t b_t^i \qquad \forall\ t, s^t$$

Use of funds:

$$c_t^i(s^t) + b_{t+1}^i + P_t(s^t) z_{t+1}^i + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1}|s^t)$$

# Lucas Tree From Class

Household Probem:

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t))\pi(s^t)$$

s.t.

$$z_t^i\big(d(s_t) + P_t(s^t)\big) + a_t^i(s^t) + R_t b_t^i \geq c_t^i(s^t) + b_{t+1}^i + P_t(s^t) z_{t+1}^i + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1}|s^t)$$

$$\forall t, s^t$$

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$$\forall t, s^t$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \Big\{ u(c_t^i(s^t))\pi(s^t) +$$

$$\lambda_t^i(s^t) \Big[ z_t^i(d(s_t) + P_t(s^t)) + a_t^i(s^t) + R_t b_t^i - c_t^i(s^t) - b_{t+1}^i - P_t(s^t) z_{t+1}^i - \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1}|s^t) \Big] \Big\}$$

# Lucas Tree from Class

* Every $s^t$ can be written as $s^t = (s_0, \dots, s_{t-1}, s_t)$
so $s^{t+1} = (s_0, \dots, s_{t-1}, s_t, s_{t+1}) = (s^t, s_{t+1})$
$\Rightarrow s^{t+1} = (s_{t+1}, s^t)$
$a^i_{t+1}(s^{t+1}) = a^i_{t+1}(s_{t+1}, s^t)$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \Big\{ u(c^i_t(s^t))\pi(s^t) +$$

$$\lambda^i_t(s^t)\Big[z^i_t(d(s_t) + P_t(s^t)) + a^i_t(s^t) + R_t b^i_t - c^i_t(s^t) - b^i_{t+1} - P_t(s^t) z^i_{t+1} - \sum_{s_{t+1}} a^i_{t+1}(s_{t+1}, s^t) Q_t(s_{t+1}|s^t)\Big]\Big\}$$

**Optimality conditions:**

$\{c^i_t(s^t)\}$: $\beta^t u'(c^i_t(s^t))\pi(s^t) - \beta^t \lambda^i_t(s^t) = 0 \Rightarrow \lambda^i_t(s^t) = u'(c^i_t(s^t))\pi(s^t)$

$\{b^i_{t+1}\}$: $-\beta^t \lambda^i_t(s^t) + \beta^{t+1} \sum_{s^{t+1}} \lambda^i_{t+1}(s^{t+1}) R_{t+1} = 0 \Rightarrow \lambda^i_t(s^t) = \beta R_{t+1} \sum_{s^{t+1}} \lambda^i_{t+1}(s^{t+1})$

$\{z^i_{t+1}\}$: $-\beta^t \lambda^i_t(s^t) P_t(s^t) + \beta^{t+1} \sum_{s^{t+1}} \lambda^i_{t+1}(s^{t+1})(d(s_{t+1}) + P_{t+1}(s^{t+1})) = 0$

$\Rightarrow \lambda^i(s^t) P_t(s^t) = \beta \sum_{s^{t+1}} \lambda^i_{t+1}(s^{t+1})(d(s^{t+1}) + P_{t+1}(s^{t+1}))$

$\{a^i_{t+1}(s_{t+1}, s^t)\}$: $-\beta^t \lambda^i_t(s^t) Q_t(s_{t+1}|s^t) + \beta^{t+1} \lambda^i_{t+1}(s^{t+1})\ 1 = 0$

$\lambda^i_t(s^t) Q_t(s_{t+1}|s^t) = \beta \lambda^i_{t+1}(s^{t+1})$

If I'm not sure of what's going on: expand summation !!

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \left\{ u(c_t^i(s^t))\pi(s^t) + \lambda_t^i(s^t) \left[ z_t^i(d(s_t) + P_t(s^t)) + a_t^i(s^t) + R_t b_t^i - c_t^i(s^t) - b_{t+1}^i - P_t(s^t) z_{t+1}^i - \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1}|s^t) \right] \right\}$$

First I'll expand the time sum $\mathcal{L} = \sum_{t=0}^{\infty} (\ell_t) = \ell_0 + \cdots + \ell_t + \ell_{t+1} + \cdots$

$$\mathcal{L} = \cdots \beta^t \sum_{s^t} u(c_t^i(s^t))\pi(s^t) +$$

$$\beta^t \sum_{s^t} \lambda_t^i(s^t) \left[ z_t^i(d(s_t) + P_t(s^t)) + a_t^i(s^t) + R_t b_t^i - c_t^i(s^t) - b_{t+1}^i - P_t(s^t) z_{t+1}^i - \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1}|s^t) \right]$$

$$+ \beta^{t+1} \sum_{s^{t+1}} u(c_{t+1}^i(s^{t+1}))\pi(s^{t+1}) +$$

$$\beta^{t+1} \sum_{s^{t+1}} \lambda_{t+1}^i(s^{t+1}) \left[ z_{t+1}^i(d(s_{t+1}) + P_{t+1}(s^{t+1})) + a_{t+1}^i(s^{t+1}) + R_{t+1} b_{t+1}^i - c_{t+1}^i(s^{t+1}) - b_{t+2}^i - P_{t+1}(s^{t+1}) z_{t+1}^i - \sum_{s_{t+2}} a_{t+2}^i(s_{t+2}, s^{t+1}) Q_{t+1}(s_{t+2}|s^{t+1}) \right] + \cdots$$

Now I can take derivatives w.r.t. $a_{t+1}^i(s_{t+1}, s^t)$, $b_{t+1}^i$ and $z_{t+1}^i$ easier

I can go on and expand $\sum_{s^{t+1}}$ to see $a_{t+1}^i(s_{t+1}, s^t)$ appears only once too !

# PS2 Problem E: Risk Sharing

Consider an economy with two agents, A and B, each receiving an endowment of consumption good every period according to the following processes:

$$\text{Agent A}: \quad y_t^A(s_t) = 1 + s_t, \quad \text{and} \quad \text{Agent B}: \quad y_t^B(s_t) = 1 - \sigma s_t,$$

where $\sigma \in [0,1]$, and $s_t$ can take two values, $s_t = 0$ with probability $\pi$, and $s_t = 1$ with probability $1 - \pi$. The consumption good received each period cannot be stored, so if not consumed it perishes.
Both agents have logarithmic utility functions and maximize the expected present value of lifetime utility denoted by:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \ln c_t^i(s^t)\pi(s^t), \quad \beta \in (0,1) \quad \text{for} \quad i = A, B.$$

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(Skipped) **Part I: Arrow-Debreu Economy (a), (b)**

In this part, we consider an economy with a complete set of Arrow-Debreu securities available for trading before time 0.

# PS2 E Part II: Lucas Tree Economy

In this part, we consider an economy with only two assets. Suppose now that the stochastic endowment of each agent, $y_t^1(s_t)$ and $y_t^2(s_t)$ above, represent the fruit dropped by a tree—Tree 1 for agent A, and Tree 2 for agent B. The two assets in the economy are the claims (i.e., stocks) issued by the two agents on the fruit dropped by their respective tree. The total claims issued for each tree are normalized to 1.
Denote the holdings of stocks on Tree 1 by agent $i$ at the beginning of time $t$ by $z^i_{1,t}$, and for Tree 2 by $z^i_{2,t}$. As usual, we assume that stock holdings cannot be negative. No other assets are available in this economy.

**(c)** Write the Lagrangian associated with the maximization problem of agent $i$ in this economy and derive the associated optimality conditions.

# PS2 E Part II: Lucas Tree Economy

**(c)** Write the Lagrangian and derive optimality conditions

Household Problem:

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \ln c_t^i(s^t)\pi(s^t), \qquad \text{for} \quad i = A, B.$$

s.t. SOURCES $\geq$ USES $\forall t, s^t$

BC : $z_{1t}^i \left( y_t^1(s_t) + p_t^1(s^t) \right) + z_{2t}^i \left( y_t^2(s_t) + p_t^2(s^t) \right) \geq z_{1t+1}^i p_t^1(s^t) + z_{2t+1} p_t^2(s^t) + c_t^i(s^t)$

SOURCES : $z_{1t}^i \left( y_t^1(s_t) + p_t^1(s^t) \right) + z_{2t}^i \left( y_t^2(s_t) + p_t^2(s^t) \right)$

USES : $z_{1t+1}^i p_t^1(s^t) + z_{2t+1} p_t^2(s^t) + c_t^i(s^t)$

# PS2 E Part II: Lucas Tree Economy

**(c)** Write the Lagrangian and derive optimality conditions

Household Problem:

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s.t. $\quad z_{1t}^i\,(y_t^1(s_t) + p_t^1(s^t)) + z_{2t}^i\,(y_t^2(s_t) + p_t^2(s^t)) \geq z_{1t+1}^i\, p_t^1(s^t) + z_{2t+1}\, p_t^2(s^t) + c_t^i(s^t) \qquad \forall t, s^t$

Lagrangian:

$$\mathcal{L} : \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \left\{ \ln c_i^t(s^t)\,\pi(s^t) + \lambda_t^i(s^t)\left[ z_{1t}^i(y_t^1(s_t) + p_t^1(s^t)) + z_{2t}^i(y_t^2(s_t) + p_t^2(s^t)) - z_{1t+1}^i\, p_t^1(s^t) - z_{2t+1}\, p_t^2(s^t) - c_t^i(s^t) \right] \right\}$$

## PS2 E Part II (c) Optimality Conditions

**(c)** Write the Lagrangian and derive optimality conditions

Lagrangian:

$$\mathcal{L}: \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \Big\{ \ln c_t^i(s^t)\,\pi(s^t) + \lambda_t^i(s^t)\Big[ z_{1t}^i\big(y_t^1(s_t)+p_t^1(s^t)\big) + z_{2t}^i\big(y_t^2(s_t)+p_t^2(s^t)\big) - z_{1t+1}^i p_t^1(s^t) - z_{2t+1} p_t^2(s^t) - c_t^i(s^t)\Big]\Big\}$$

$\{c_t^i(s^t)\}$ $$\beta^t \frac{1}{c_t^i(s^t)}\pi(s^t) - \beta^t \lambda_t^i(s^t) = 0 \Rightarrow \lambda_t^i(s^t) = \frac{\pi(s^t)}{c_t^i(s^t)}$$

$\{z_{1t+1}^i\}$ $$-\beta^t \lambda_t^i(s^t)\, p_t^1(s^t) + \beta^{t+1}\sum_{s^{t+1}} \lambda_{t+1}^i(s^{t+1})\big(y_{t+1}^1(s_{t+1}) + p_{t+1}^1(s^{t+1})\big) = 0$$

$$\Rightarrow 1 = \beta \sum_{s^{t+1}} \frac{\lambda_{t+1}^i(s^{t+1})}{\lambda_t^i(s^t)} \frac{\big(y_{t+1}^1(s_{t+1}) + p_{t+1}^1(s^{t+1})\big)}{p_t^1(s^t)}$$

$\{z_{2t+1}^i\}$ $$-\beta^t \lambda_t^i(s^t)\, p_t^2(s^t) + \beta^{t+1}\sum_{s^{t+1}} \lambda_{t+1}^i(s^{t+1})\big(y_{t+1}^2(s_{t+1}) + p_{t+1}^2(s^{t+1})\big) = 0$$

$$\Rightarrow 1 = \beta \sum_{s^{t+1}} \frac{\lambda_{t+1}^i(s^{t+1})}{\lambda_t^i(s^t)} \frac{\big(y_{t+1}^2(s_{t+1}) + p_{t+1}^2(s^{t+1})\big)}{p_t^2(s^t)}$$

# Gift : PS2 E II Part (d)

(d) $\boxed{\text{From FOC, show FTAP holds : } 1 = E(mR)}$

$\{z^i_{1t+1}\}$
$$1 = \beta \sum_{s^{t+1}} \frac{\lambda^i_{t+1}(s^{t+1})}{\lambda^i_t(s^t)} \frac{\left(y^1_{t+1}(s_{t+1}) + P^1_{t+1}(s^{t+1})\right)}{P^1_t(s^t)}$$

$\{z^i_{2t+1}\}$
$$1 = \beta \sum_{s^{t+1}} \frac{\lambda^i_{t+1}(s^{t+1})}{\lambda^i_t(s^t)} \frac{\left(y^2_{t+1}(s_{t+1}) + P^2_{t+1}(s^{t+1})\right)}{P^2_t(s^t)}$$

$\{c^i_t(s^t)\}$
$$\frac{\lambda^i_{t+1}(s^{t+1})}{\lambda^i_t(s^t)} = \frac{\dfrac{\pi(s^{t+1})}{c^i_{t+1}(s^{t+1})}}{\dfrac{\pi(s^t)}{c^i_t(s^t)}} = \frac{\pi(s^{t+1})}{\pi(s^t)} \frac{c^i_t(s^t)}{c^i_{t+1}(s^{t+1})} = \pi(s_{t+1}|s^t) \frac{c^i_t(s^t)}{c^i_{t+1}(s^{t+1})}$$

$$\pi(s^{t+1}) = \pi(s_{t+1}, s_t, s_{t-1}, \dots, s_0)$$
$$\pi(s^t) = \pi(s_t, s_{t-1}, \dots, s_0)$$

By conditional pr :

$$\pi(s_{t+1}, s_t, \dots, s_0) = \pi(s_{t+1} | s_t, \dots, s_0) \cdot \pi(s_t, \dots, s_0)$$

$$\Rightarrow \frac{\pi(s^{t+1})}{\pi(s^t)} = \pi(s_{t+1} | s_t, \dots, s_0) = \pi(s_{t+1} | s^t) \,//$$

## Plugging back

$$\{z^i_{1t+1}\} \qquad 1 = \sum_{s^{t+1}} \pi(s_{t+1}|s^t)\, \underbrace{\beta \frac{c^i(s^t)}{c^i_{t+1}(s^{t+1})}}_{m_{t+1}(s^{t+1})} \underbrace{\frac{(y^1_{t+1}(s_{t+1}) + p^1_{t+1}(s^{t+1}))}{p^1_t(s^t)}}_{R_{1,t+1}(s^{t+1})}$$

$$\{z^i_{2t+1}\} \qquad 1 = \underbrace{\sum_{s^{t+1}} \pi(s_{t+1}|s^t)}\, \underbrace{\beta \frac{c^i(s^t)}{c^i_{t+1}(s^{t+1})}}_{m_{t+1}(s^{t+1})} \underbrace{\frac{(y^2_{t+1}(s_{t+1}) + p^2_{t+1}(s^{t+1}))}{p^2_t(s^t)}}_{R_{2,t+1}(s^{t+1})}$$

this is a sum in which I multiply each term $X(s)$ times the probability that "s" occurs = An expectation!

$$E(X) = \sum_s X(s) \cdot Pr(s)$$

$$\Rightarrow \{z^i_{1t+1}\} \qquad 1 = E_t\left[m_{t+1} R_{1,t+1}\right] \text{//}$$

$$\{z^i_{2t+1}\} \qquad 1 = E_t\left[m_{t+1} R_{2,t+1}\right] \text{//}$$

FTAP //

# EXTRA !!

**For the curious :**

Its not mandatory to put the constraint inside the $\beta$. But it <u>needs</u> to be under $\sum_{t=0}^{\infty}$ and $\sum_{s^t}$ !

Because Budget constraint holds $\forall\, t, s^t$ !!

$$\max \quad \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \ln c_t^i(s^t)\, \pi(s^t)$$

$$\text{s.t.} \quad z_{1,t}^i\,(y_t^1(s^t) + p_t^1(s^t)) + z_{2,t}^i\,(y_t^2(s^t) + p_t^2(s^t)) \geq p_t^1(s^t)\, z_{1,t+1}^i + p_t^2(s^t)\, z_{2,t+1}^i + c_t^i(s^t) \qquad \forall\, t, s^t$$

$$\mathcal{L}: \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \ln c_t^i(s^t)\, \pi(s^t) + \sum_{t=0}^{\infty} \sum_{s^t} \lambda^i(s^t)\Big(z_{1,t}^i\,(y_t^1(s^t) + p_t^1(s^t)) + z_{2,t}^i\,(y_t^2(s^t) + p_t^2(s^t)) - p_t^1(s^t)\, z_{1,t+1}^i - p_t^2(s^t)\, z_{2,t+1}^i - c_t^i(s^t)\Big)$$

$$\{c_t^i(s^t)\}: \quad \frac{\beta^t\, \pi(s^t)}{c_t^i} = \lambda^i(s^t)$$

$$\{z_{1,t+1}^i\}: \quad \lambda^i(s^t)\, p_t^1(s^t) = \sum_{s^{t+1}} \lambda^i(s^{t+1})\left[y_{t+1}^1(s^t) + p_{t+1}^1(s^{t+1})\right] \Rightarrow \quad 1 = \sum_{s^{t+1}} \frac{\lambda^i(s^{t+1})}{\lambda^i(s^t)}\, \frac{y_{t+1}^1(s^t) + p_{t+1}^1(s^{t+1})}{p_t^1(s^t)}$$

$$\{z_{2,t+1}^i\}: \quad \lambda^i(s^t)\, p_t^2(s^t) = \sum_{s^{t+1}} \lambda^i(s^{t+1})\left[y_{t+1}^2(s^t) + p_{t+1}^2(s^{t+1})\right] \Rightarrow \quad 1 = \sum_{s^{t+1}} \frac{\lambda^i(s^{t+1})}{\lambda^i(s^t)}\, \frac{y_{t+1}^2(s^t) + p_{t+1}^2(s^{t+1})}{p_t^2(s^t)}$$

# PS2 Problem D: Lucas Tree and Storage Technology (Midterm 2024)

Consider an economy populated by two agents, A and B. Each agent owns a tree that drops fruit every period according to the stochastic processes:

$$\text{Agent A owns Tree 1, fruit:} \quad y_t^A(s_t) = 1 + s_t,$$
$$\text{Agent B owns Tree 2, fruit:} \quad y_t^B(s_t) = 1 - s_t,$$

where $s_t$ can take two values, $s_t = 0$ with probability 0.5, and $s_t = 1$ with probability 0.5.

The fruit received each period can be stored using a storage technology so that for each unit stored, the agent gets $\gamma$ units of fruit next period, with $\gamma \in (0, 1)$.

Both agents have logarithmic utility functions and maximize the expected present value of lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \ln c_t^i(s^t)\pi(s^t), \quad \beta \in (0, 1), \quad \text{for} \quad i = A, B.$$

# PS2 Problem D: Assets and Market Structure

The state $s_0$ is unrealized when agents solve their maximization problem, so $s_0 = 0$ with probability 0.5, and $s_0 = 1$ with probability 0.5.
There are two assets in the economy representing claims (i.e., stocks) issued by the two agents on the fruit dropped by their respective tree. The total claims issued for each tree are normalized to 1.

- Denote the holdings of stocks on Tree 1 by agent $i$ at the beginning of time $t$ by $z^i_{1,t}$, and for Tree 2 by $z^i_{2,t}$.
- The assets are traded in perfectly competitive asset markets at the ex-dividend prices $P^t_1(s^t)$ and $P^t_2(s^t)$, respectively.
- Finally, denote the fruit stored by agent $i$ in period $t$ to be carried into period $t+1$ by $k^i_{t+1}$, where $k^i_{t+1} \geq 0$.

# PS2 Problem D: (a) Budget Constraint

**(a)** Write the per-period budget constraint of the agents in this economy. The budget constraint for agent $i$ at time $t$ is:

$$z^i_{1,t}\left(y^1_t(s_t)+p^1_t(s^t)\right)+z^i_{2,t}\left(y^2_t(s_t)+p^2_t(s^t)\right)+\gamma k^i_t \geq c^i_t(s^t)+k^i_{t+1}+z^i_{1,t+1}p^2_{t+1}(s^t)+z^i_{2,t+1}p^2_{t+1}(s^t) \quad \forall\, s^t, t$$

where $y^j_t(s_t)$ represents the dividend payout from Tree $j$. The agent can use income from endowment, asset returns, and storage to finance consumption and asset purchases.

USES: $c^i_t(s^t)+k^i_{t+1}+z^i_{1,t+1}p^2_{t+1}(s^t)+z^i_{2,t+1}p^2_{t+1}(s^t)$

SOURCES: $z^i_{1,t}\left(y^1_t(s_t)+p^1_t(s^t)\right)+z^i_{2,t}\left(y^2_t(s_t)+p^2_t(s^t)\right)+\gamma k^i_t$

# PS2 Problem D: (b) Lagrangian and Optimality Conditions

**(b)** Write the Lagrangian for agent $i$ and obtain the corresponding optimality conditions.

**Hint:** Make sure to include the non-negativity constraint on $k^i_{t+1} \geq 0$; you can ignore the transversality conditions.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Bigg\{ \sum_{s^t} \ln c^i_t(s^t)\pi(s^t) + \lambda^i_t(s^t) \Bigg[ z^i_t\big(P^1_t(s^t) + y^1_t(s_t)\big)$$

$$+ z^i_2 t\big(P^2_t(s^t) + y^2_t(s_t)\big) + \gamma k^i_t$$

$$- c^i_t(s^t) - z^i_{t+1} P^1_t(s^t) - z^i_{2t+1} P^2_t(s^t) - k^i_{t+1} \Bigg] + \theta^i_t(s^t) k^i_{t+1} \Bigg\}$$

# PS2 Problem D: (b) Optimality Conditions

The optimality conditions are

$$\{c^i(s^t)\} \quad \frac{1}{c_t^i(s^t)}\pi(s^t) = \lambda_t^i(s^t)$$

$$\{z_{1,t+1}^i\} \quad -\lambda_t^i(s^t)P_t^1(s^t) + \beta \sum_{s_{t+1}} \left[P_{t+1}^1(s^{t+1}) + y_{t+1}^1(s_{t+1})\right]\lambda_{t+1}^i(s^{t+1}) = 0$$

$$\{z_{2,t+1}^i\} \quad -\lambda_t^i(s^t)P_t^2(s^t) + \beta \sum_{s_{t+1}} \left[P_{t+1}^2(s^{t+1}) + y_{t+1}^2(s_{t+1})\right]\lambda_{t+1}^i(s^{t+1}) = 0$$

$$\{k_{t+1}^i\} \quad -\lambda_t^i(s^t) + \beta\gamma \sum_{s_{t+1}} \lambda_{t+1}^i(s^{t+1}) + \theta_t^i(s^t) = 0$$

CS $$\{\theta^i(s^t)\} \quad \theta_t^i(s^t)k_{t+1}^i = 0, \quad \theta_t^i(s^t) \geq 0.$$

for $i = A, B$, and for all $t$'s and $s^t$'s.

# PS2 Problem D: (c) Market Clearing and Risk Sharing

**(c)** State the market clearing conditions for the economy.

Market clearing for fruit (output)

$$\sum_{i=A,B} c_t^i(s^t) + \sum_{i=A,B} k_{t+1}^i = y^t(s^t) + \gamma \sum_{i=A,B} k_t^i, \quad \forall t, s^t$$

Market clearing for claims on trees:

$$\sum_{i=A,B} z_{1,t}^i = 1, \quad \sum_{i=A,B} z_{2,t}^i = 1.$$

**Evaluation:** Can you evaluate if the agents are able to fully insure and obtain consumption that does not depend on $s_t$? If yes, why? If not, why not?

**Hint:** Guess a consumption path independent of $s_t$ and check whether the optimality conditions and the market clearing conditions are satisfied.

# Useful Summaries

# Arrow Debreu Date 0 Economy

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t))\pi(s^t)$$

$$\sum_{t=0}^{\infty} \sum_{s^t} c_t^i(s^t) q_t^0(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} y_i^t(s^t) q_t^0(s^t)$$

From FONC:

$$q_t^0(s^t) = \beta^t \frac{u'(c_t^i(s^t))}{\lambda_i} \pi(s^t) \quad \forall s^t, t, i$$

An instance of Fundamental Theorem of Asset Pricing!

$$P_t^i = E_t\Big[\beta \frac{u'(c_{t+1}(s))}{u'(c_t)} x_{t+1}^i(s)\Big]$$

In fact, you can use $q_t^0(s^t)$ to price any asset:

$$p^0 = \sum_t \sum_{s^t} q_t^0(s^t) x_t(s^t)$$

# Fundamental Theorem of Asset Pricing

$$P^i = E[m(s)x^i(s)]$$

$$P_t^i = E_t[m_{t+1}(s)x_{t+1}^i(s)]$$

$$P_t^i = E_t\left[\beta \frac{u'(c_{t+1}(s))}{u'(c_t)} x_{t+1}^i(s)\right]$$

# Asset Pricing: Summary

Three cases in discrete time:

- Arrow Debreu Markets:
  - All trade happens at time zero
  - Prices of AD securities are $q_t^0(s^t)$
- Arrow Sequential Markets:
  - Each period you trade securities to cover the following period risk
  - Prices of AS securities (given time $t$ and $s^t$) are $Q_t(s_{t+1} \mid s^t)$
- Lucas Tree Economy
  - You trade at each period looking one period ahead
  - At least you will have one asset. A "tree" $z_t$ that pays dividends $d_t(s^t)$ and has a price $P_t$.
  - You might also have more (different) trees, bonds, AS securities.

Useful:

- Fundamental theorem of asset pricing:
  $P = E[m(s)x(s)] = E[m]E[x] + cov[m, x]$
- Equivalence between AD and AS: $q_{t+1}^0(s^{t+1}) = Q_t(s_{t+1} \mid s^t)q_t^0(s^t)$
- Pricing of an asset: $p^0 = \sum_t \sum_{s^t} q_t^0(s^t)x_t(s^t)$