PS1 Problem E: Investment with Adjustment Costs

(d) Use the two equation you identified in (c) to build a phase diagram for the dynamics of the system with k on the horizontal axis and q on the vertical axis. Place the loci in the diagram and specify how the system moves in each quadrant of the diagram. Finally, argue where the stable manifold lies, and show the path that the system would follow for an initial capital level that is lower than the steady state.

1. **Loci**: graph the points (k, q) at which (1) $\dot{q}(t) = 0$ and (2) $\dot{k}(t) = 0$.

$$\dot{q}(t) = 0 \iff (1) \quad q(t) = \frac{f'(k(t))}{r + \delta}$$
$$\dot{k}(t) = 0 \iff (2) \quad q(t) = \alpha\delta k(t) + 1$$

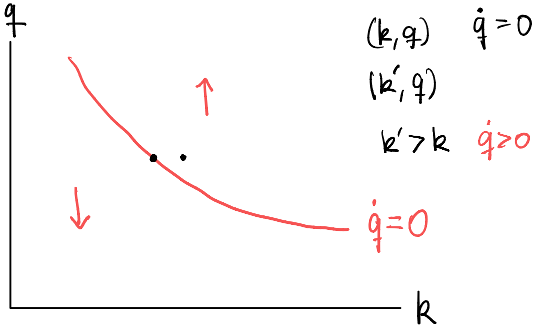
2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

PS1 Problem E: Phase Diagram

1. **Loci**: graph the points (k, q) at which (1) $\underline{\dot{q} = (r+\delta)q - f'(k)} = 0$

$$\dot{q} = 0 \iff (1) \quad q = \frac{f'(k)}{r+\delta}$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

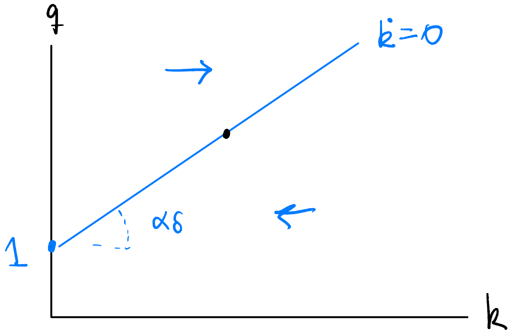


PS1 Problem E: Phase Diagram

1. **Loci**: graph the points (k, q) at which (2) $\dot{k} = \frac{1}{\alpha}(q-1) - \delta k = 0$.

$$\dot{k} = 0 \iff (2) \quad \underline{q = \alpha\delta k + 1}$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2)



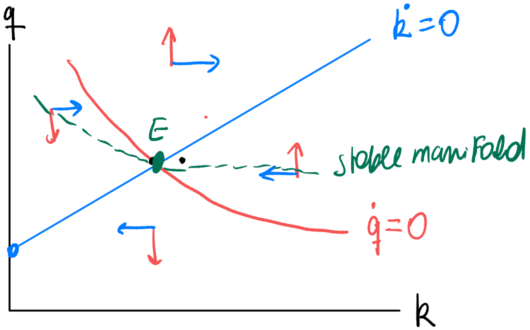
$(k, q) \Rightarrow \dot{k} = 0$

$(k, q') \Rightarrow \dot{k} > 0$

$q' > q$

PS1 Problem E: Phase Diagram

3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.



PS1 Problem F: Modeling Fertility Choice in Cts Time

(b) Consider the following statement: *"Introducing preferences for fertility is similar to modeling an infinite-horizon consumer that can choose its degree of impatience."* Please discuss the statement in the context of the model above. [Hint: solve for the differential equation for $N(t)$ and think about how $n(t)$ enters in the preferences of the head of the dynasty.]

preferences: $$\int_0^\infty e^{-\rho t} c(t)^{1-\eta} N(t)^{1-\varepsilon} dt,$$

ODE for N(t): $\dot{N}(t) = n(t)N(t)$

$\dot{N}(t) - n(t)N(t) = 0$

solution will be $N(t) = e^{\int_0^t n(s)ds} N_0$ → proof for this in 2 slides, please first read proof for ★ in next slide

Note this is a more general version of the ODE we've been dealing with:

$\dot{X}(t) + rX(t) = 0$: $X(t) = e^{-rt} X_0$

$\dot{X}(t) - rX(t) = 0$: $X(t) = e^{rt} X_0$

↓ ★ proof for this in next slide

<u>Note</u>: $\int_0^t r\,ds = rs|_0^t = r(t-0) = rt$

$\Rightarrow e^{\int_0^t r ds} = e^{rt}$ //

ODE : $\dot{x}(t) - r\,x(t) = 0$ $\Longrightarrow$ solution: $x(t) = e^{rt} x_0$

"Proof"

1. Note $\dfrac{d\, e^{-rt} x(t)}{dt} = e^{-rt}\left(\dot{x}(t) - r\,x(t)\right)$

2. $\dot{x}(t) - r\,x(t) = 0 \qquad / \cdot e^{-rt}$

$$e^{-rt}\left(\dot{x}(t) - r\,x(t)\right) = 0 \qquad / \int dt \quad \text{indefinite integral, i.e. antiderivative}$$

$$\int \frac{d e^{-rt} x(t)}{dt}\, dt = \int 0\, dt$$

$$e^{-rt} x(t) + A = B$$

$$e^{-rt} x(t) = C \qquad : C = B - A$$

$$x(t) = e^{rt} C$$

3. $x(0)$: $x(0) = e^{0} C = C = x_0$

4. $x(t) = e^{rt} x_0$ //

ODE : $\dot{N}(t) - n(t) N(t) = 0 \implies$ Solution : $N(t) = e^{\int_0^t n(s) ds} N_0$

" Proof " : ① Note $\frac{d \, e^{-\int_0^t n(s) ds} N(t)}{dt} = e^{-\int_0^t n(s) ds} [\dot{N}(t) - n(t) N(t)]$

why? $\frac{d}{dt} \int_0^t n(s) ds = n(t)$!!

then $\frac{d}{dt} \left(e^{-\int_0^t n(s) ds} \right) = e^{-\int_0^t n(s) ds} \cdot n(t) \cdot -1$

② $\dot{N}(t) - n(t) N(t) = 0 \quad / \cdot \; e^{-\int_0^t n(s) ds}$

$$\frac{d\left(e^{-\int_0^t n(s) ds} N(t) \right)}{dt} = 0 \quad / \cdot \int \; dt$$

$$e^{-\int_0^t n(s) ds} N(t) + A = B$$

... (2) $$N(t) = e^{\int_0^t n(s)\,ds}\, C$$

(3) $$N(0) = e^0\, C = C = N_0$$

(4) $$N(t) = e^{\int_0^t n(s)\,ds}\, N_0 \quad //$$