Econ 210B
Discussion Session N°4

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Plan for Today

PS1 Problem E: Investment with Adjustment Costs

PS1 Problem F: Modeling Fertility Choice in Continuous Time (Midterm 2024)

Casting the Neoclassical Growth Model in HJB*

PS1 Problem E: Investment with Adjustment Costs

Consider the problem of an infinitely lived firm that is producing output using capital, $k(t)$, according to the production function, $f(k(t))$, where f is strictly increasing, strictly concave, and twice continuously differentiable, and $f(0) = 0$.

The firm chooses investment $x(t)$ in order to maximize the present discounted value of profits, where the discount factor is e^{-rt}, for some real interest rate $r > 0$. When investing $x(t)$ the firm is facing adjustment costs equal to $\frac{1}{2}\alpha x(t)$ per unit of investment, where $\alpha > 0$.

The objective function of the firm can be thus written as

$$\int_0^\infty e^{-rt} \left[f(k(t)) - \left(1 + \frac{1}{2}\alpha x(t) \right) x(t) \right] dt$$

The law of motion for capital is

$$\dot{k}(t) = x(t) - \delta k(t)$$

where $\delta > 0$ and the initial capital level is $k_0 > 0$

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(0) Indicate the state and control variables.

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In short:

(a) Write down the current value Hamiltonian associated with the problem. Obtain optimality conditions.

(b) Identify an investment equation, relating $x(t)$ and $q(t)$. Interpret, providing economic intuition.

(c) Find an equation for $\dot{q}(t)$ and $\dot{k}(t)$. Characterize the steady state. What happens when α increases? Interpret.

(d) Build a Phase Diagram.

PS1 Problem E: Investment with Adjustment Costs

(a) Write the current value Hamiltonian associated with the optimization problem for the firm, and indicate the state variable and the control variable. Please denote the co-state variable in your problem by $q(t)$. Obtain the optimality conditions that characterize the solution to the firm's problem. Make sure to include all of them.

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Optimality conditions:

$$\begin{aligned}
(x) \quad & 1 + \alpha x(t) = q(t) \\
(k) \quad & \dot{q}(t) - rq(t) = -\left(f'(k(t)) - \delta q(t)\right) \\
(q) \quad & \dot{k}(t) = x(t) - \delta k(t)
\end{aligned}$$

$$(TC) \quad \lim_{t \to \infty} e^{-rt} q(t) k(t) = 0$$

PS1 Problem E: Investment with Adjustment Costs

(b) Among the optimality conditions identify an "investment" equation that relates the value of $q(t)$ to $x(t)$.
According to such investment equation, when should the investment rate be positive, whenshould it be negative? Using the interpretation of the co-state variable that we have often used in class, provide an economic intuition for your answer. What happens to the investment rate when adjustment costs are more severe, i.e. α is larger? What happens to the investment rate when $\alpha \to 0$?
In light of your answer to this last question, why do you think that dynamic models with adjustment cost are widely used in macroeconomics?

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- From (x) $\quad 1 + \alpha x(t) = q(t)$

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- Larger $\alpha \implies$ for any given $q(t)$, $x(t)$ is smaller

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- Larger $\alpha \implies$ for any given $q(t)$, $x(t)$ is smaller
- $\lim_{\alpha\to 0} \frac{1}{\alpha}(q(t)-1) = \infty$ implying k jumps immediately to optimal

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  Recall the constrained investment problem. If we started from $k(0) = k_0$, we set $I(t) = \bar{I}$ to get as fast as possible to k^{ss}. But if no upper constraint on investment, what is the optimal $I(t)$?

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- Solution: **q-Theory**, Convex adjustment cost
  - Installing capital is costly
  - Capital does not adjust instantly but gradually

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- But then ... does capital adjust smoothly? More to come in 210C!

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(c) Obtain a representation of the dynamic system for the optimal solution to the firm's problem in terms of two equations, one for $\dot{q}(t)$, and the other for $\dot{k}(t)$.
Use these equations to characterize the steady state of the system (defined as the point where $\dot{q}(t) = \dot{k}(t) = 0$) How does capital in the steady state change when adjustments costs are more severe, i.e. α is larger? Please provide a formal argument and an economic intuition.

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$$\dot{q}(t) = 0 \iff q(t) = \frac{f'(k(t))}{r+\delta}$$

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Both in steady state: $\quad q^* = \frac{f'(k^*)}{r+\delta} \qquad k^* = \frac{1}{\alpha\delta}(q^*-1)$

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- Larger $\alpha \implies$ smaller k^*

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  - Adjustment costs in steady state are $\alpha\delta k^*$
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  - Adjustment costs in steady state are $\alpha\delta k^*$
  - $f'(k^*)$ needs to be larger, k^* smaller
- If no adjustment cost ($\alpha \to 0$), $q(t) = 1$, $f'(k(t)) = r + \delta$
  - Capital jumps immediately to the steady state and $x(0) = \infty$

PS1 Problem E: Investment with Adjustment Costs

(d) Use the two equation you identified in (c) to build a phase diagram for the dynamics of the system with k on the horizontal axis and q on the vertical axis. Place the loci in the diagram and specify how the system moves in each quadrant of the diagram. Finally, argue where the stable manifold lies, and show the path that the system would follow for an initial capital level that is lower than the steady state.

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1. **Loci**: graph the points (k, q) at which (1) $\dot{q}(t) = 0$ and (2) $\dot{k}(t) = 0$.

$$\dot{q}(t) = 0 \iff (1) \quad q(t) = \frac{f'(k(t))}{r + \delta}$$
$$\dot{k}(t) = 0 \iff (2) \quad q(t) = \alpha\delta k(t) + 1$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) <u>separately</u>.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

PS1 Problem E: Phase Diagram

1. **Loci**: graph the points (k, q) at which (1) $\dot{q} = (r + \delta)q - f'(k) = 0$

$$\dot{q} = 0 \quad \Longleftrightarrow \quad (1) \quad q = \frac{f'(k)}{r + \delta}$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

PS1 Problem E: Phase Diagram

1. **Loci**: graph the points (k, q) at which (2) $\dot{k} = \frac{1}{\alpha}(q-1) - \delta k = 0$.

$$\dot{k} = 0 \iff (2) \quad q = \alpha\delta k + 1$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2)

PS1 Problem E: Phase Diagram

3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

PS1 Problem F: Modeling Fertility Choice in Continuous Time (Midterm 2024)

Suppose that we are interested in modeling the relationship between productivity and fertility choice. One question we would like to ask the model is how fertility choice, and thus population growth, responds to a permanent technological improvement.
Let $N(t)$ denote the size of the population at time t. Let $n(t)$ be the instantaneous growth rate of the population, so that

$$\dot{N}(t) = n(t)N(t).$$

The simplest possible way to model fertility choice is to make $n(t)$ a choice variable in the context of an optimizing framework. At time $t = 0$ the optimizing agent is the head of a dynasty that has to decide the entire path of $n(t)$ into the future. generation in the dynasty lives only for one instant and receives utility from their own consumption and from the utility of all generations living after them.
Note that $n(t)$ is the net growth in the population (size of the dynasty). Because we are assuming that a generation only lives for one instant t, when $n(t) = 0$ each person is replaced by a new offspring in the next instant. Thus, $n(t)$ measures the fertility in excess of (or short of if negative) the level that would keep the population constant.

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Let the total utility of the head of the dynasty be specified as

$$\int_0^\infty e^{-\rho t} c(t)^{1-\eta} N(t)^{1-\varepsilon} dt,$$

where $\eta \in (0,1)$, $\rho > 0$, and $\varepsilon \in (0,1)$. Note that $N(t)^{1-\varepsilon}$ represents the extent to which the head of the dynasty cares about the size of the dynasty at the instant t. The term $c(t)$ in the utility function above represents consumption per capita, that is, consumption per member of the dynasty.

Total output in the economy is assumed to be

$$Y(t) = AN(t),$$

where $A > 0$. Consumption per capita is

$$c(t) = \frac{Y(t)}{N(t)} - \phi n(t),$$

where $\phi > 0$. The term $\phi n(t)$ in the consumption equation assumes that any increase in population subtracts resources from the population living in instant t. It captures the idea that having children reduces the resources parents have available for their personal enjoyment.

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(a) Write the present value Hamiltonian associated with the optimization problem for the head of the dynasty, and indicate the state variable and the control variable(s). Please denote the co-state variable in your problem by $\lambda(t)$.

(b) Consider the following statement: *"Introducing preferences for fertility is similar to modeling an infinite-horizon consumer that can choose its degree of impatience."* Please discuss the statement in the context of the model above. [Hint: solve for the differential equation for $N(t)$ and think about how $n(t)$ enters in the preferences of the head of the dynasty.]

(c) Obtain the optimality conditions for the control, the co-state, and the state variables. Explain how you can determine the transversality condition for this problem.

(d) Provide an economic interpretation of the co-state variable in this problem. Using such interpretation, provide an economic interpretation for the optimality condition on the control variable, and for the law of motion for the co-state variable.

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- The optimization problem for the head of the dynasty is characterized by the following present value Hamiltonian:

$$\mathcal{H} = e^{-\rho t} \left(A - \phi n(t)\right)^{1-\eta} N(t)^{1-\epsilon} + \lambda(t) n(t) N(t)$$

where:

- $n(t)$ is the control variable
- $N(t)$ is the state variable
- $\lambda(t)$ is the co-state variable

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- The differential equation for $N(t)$ is:

$$N(t) = N_0 e^{\int_0^t n(s) ds}.$$

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- Substituting this into the utility function, the present value of utility flow at time t becomes:

$$e^{-\rho t + (1-\epsilon)\int_0^t n(s)ds} c(t)^{1-\eta} N_0^{1-\epsilon}.$$

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- This expression shows that the accumulated growth of population, $\int_0^t n(s)ds$, acts as an endogenous component of the discount factor. A higher population growth rate reduces the effective discount rate, making the dynasty more patient over time.

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- The maximum principle and optimality conditions for the optimization problem are:

Control variable:

$$-\phi(1-\eta)e^{-\rho t}\left(A-\phi n(t)\right)^{-\eta} N(t)^{1-\epsilon} + \lambda(t)N(t) = 0.$$

Co-state variable:

$$\dot{\lambda}(t) = -e^{-\rho t}\left(A-\phi n(t)\right)^{1-\eta}(1-\epsilon)N(t)^{-\epsilon} - \lambda(t)n(t).$$

State variable:

$$\dot{N}(t) = n(t)N(t).$$

Transversality Condition:

$$\lim_{t\to\infty} \lambda(t)N(t) = 0.$$

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- $\lambda(t)$: marginal impact of an additional unit of population $N(t)$ on the dynasty's maximized lifetime utility.

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- $\lambda(t)$: marginal impact of an additional unit of population $N(t)$ on the dynasty's maximized lifetime utility.
- The optimality condition for $n(t)$ equates the marginal cost of increasing fertility—represented by lower consumption—to the marginal benefit, captured by the impact on future population growth and its contribution to utility.

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- The optimality condition for $n(t)$ equates the marginal cost of increasing fertility—represented by lower consumption—to the marginal benefit, captured by the impact on future population growth and its contribution to utility.
- The law of motion for $\lambda(t)$ shows that its change over time is determined by the marginal value of population in terms of utility and how population growth affects the overall lifetime utility of the dynasty.

Casting the Neoclassical Model in HJB

$$\rho v(x(t)) = \max_{y(t)} \left[\tilde{g}(x(t), y(t)) + \dot{v}(x(t))\right]$$

Stationary Hamilton-Jacobi-Bellman (HJB) Equation

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$$\text{s.t.} \quad \dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

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$$(c) \quad u'(c(t)) = v_k(k(t))$$

$$(HJB_k) \quad \rho v_k = v_{kk}(k(t))(f(k(t)) - c(t) - \delta k(t)) + v_k(k(t))(f'(k(t)) - \delta)$$

$$v_{kk} k(t)\dot{k}(t) = v_k(k(t))(\rho + \delta - f'(k(t))$$