Econ 210B
Discussion Session N°3

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Plan for Today

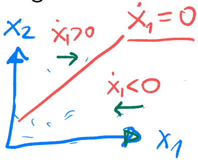
Phase Diagrams: Foundations

Example: Neoclassical Growth Model

Practice Question: PS1 D. Natural Resources

Some Definitions

- **System of Ordinary Differential Equations (ODE)**: Set of equations involving multiple interdependent variables and their rate of change to a single independent variable (typically t). Example:



(1) $\dot{x}_1 = a x_1 + b x_2$

(2) $\dot{x}_2 = c x_1 + d x_2$

$\dot{x} = A x$

(x_1, x_2)

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), \ldots, x_n(t), t)$$
$$\dot{x}_2(t) = f_2(x_1(t), x_2(t), \ldots, x_n(t), t)$$
$$\vdots$$
$$\dot{x}_n(t) = f_n(x_1(t), x_2(t), \ldots, x_n(t), t)$$

$\rightarrow$ $\dot{x} > 0$

$\leftarrow$ $\dot{x} < 0$

$\uparrow$ $\dot{y} > 0$

$\downarrow$ $\dot{y} < 0$

Where $\dot{x}_i(t) \equiv \frac{dx_i(t)}{dt}$ for each $i = 1, 2, \ldots, n$.

- **Phase Diagram**: Used to visualize the behavior of an ODE system (typically with 2 variables). Includes:
  - **Loci**: Points were one variable is at rest, i.e. $\dot{x}_i(t) = 0$.
  - **Regions**: Each locus divides the graph in 2 regions with respect to the sign of $\dot{x}_i(t)$.
  - **Winds**: To illustrate change of the variable in x-axis (y-axis) we use $\rightarrow$ $\leftarrow$ ($\downarrow$ $\uparrow$).

Building a Phase Diagram

$x(t),\ y(t)$

Think about a general 2 variable system (y and x) defined by the ODE:

$$\dot{x}(t) = ax + by + h$$
$$\dot{y}(t) = cx + dy + k$$

1. **Loci**: graph the points (x, y) at which (1) $\dot{x}(t) = 0$ and (2) $\dot{y}(t) = 0$.

$$\dot{x}(t) = 0 \quad \Longleftrightarrow ax + by + h = 0 \Longleftrightarrow y = -\frac{a}{b}x - \frac{h}{b}$$
$$\dot{y}(t) = 0 \quad \Longleftrightarrow cx + dy + k = 0 \Longleftrightarrow y = -\frac{c}{d}x - \frac{k}{d}$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

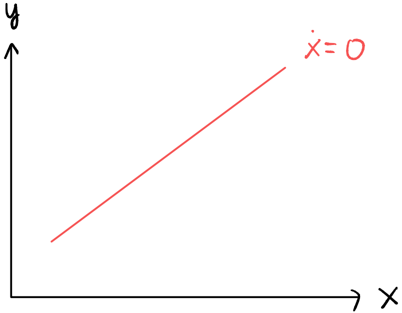
Building a Phase Diagram: (1) $\dot{x}(t) = 0$

1. **Loci**: graph the points (x, y) at which (1) $\dot{x}(t) = 0$.

$$\dot{x}(t) = 0 \iff ax + by + h = 0 \iff y = -\frac{a}{b}x - h$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

Step 1.
Just graph
$\dot{x} = 0$



slope :
$\underbrace{-\frac{a}{\underbrace{b}_{<0}}}_{>0}$

Assume $a, c, d < 0$, $b > 0$.

Building a Phase Diagram: (1) $\dot{x}(t) = 0$

1. **Loci**: graph the points (x, y) at which (1) $\dot{x}(t) = 0$.

$$\dot{x}(t) = 0 \iff ax + by + h = 0 \iff y = -\frac{a}{b}x - h$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

$\rightarrow$ Hint: Take (x, y) as given, then move y and see what happens to $\dot{x}$.

STEP 2. Graph the "winds" at each side of $\dot{x}=0$.
at one side we will have $\dot{x}>0$ $(\rightarrow)$, and at the other we will have $\dot{x}<0$ $(\leftarrow)$

How? Here's the guide I explained in the TA.

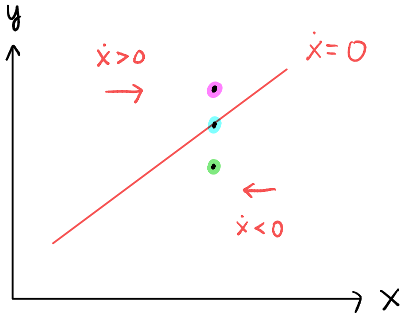
Take a point (x,y) in your locus $\dot{x}=0$

Now fixing x, choose $y'>y$
your new point is (x,y')
Now observe the ODE

at (x,y) $\dot{x}(t)=0$ $\Rightarrow$ $\dot{x}(t) = ax + by + h = 0$

Then at (x,y') $\Rightarrow$ $\dot{x}(t) = ax + by' + h > 0$

(because $b>0$)



[In TA we also did it for $y''<y$, then you get
$\dot{x}(t) = ax + by'' + h < 0$]

Assume $a, c, d < 0$, $b > 0$.

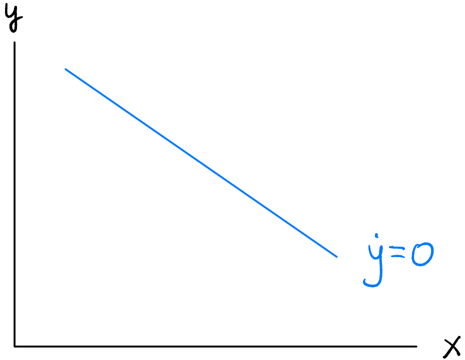
Building a Phase Diagram: (2) $\dot{y}(t) = 0$

1. **Loci**: graph the points (x, y) at which (2) $\dot{y}(t) = 0$.

$$\dot{y}(t) = 0 \iff cx + dy + k = 0 \iff y = -\frac{c}{d}x - k$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2).

again
step 1,
now for $\dot{y}=0$



Assume $a, c, d < 0,\ b > 0$.

Building a Phase Diagram: (2) $\dot{y}(t) = 0$

1. **Loci**: graph the points (x, y) at which (2) $\dot{y}(t) = 0$.

$$\dot{y}(t) = 0 \iff cx + dy + k = 0 \iff y = -\frac{c}{d}x - k$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2).

$\rightarrow$ Hint: Take (x, y) as given, then move x and see what happens to $\dot{y}$.

Step 2 for $\dot{y}=0$

Again Take (x, y)

where $\dot{y}(t)=0$,

so $\dot{y}(t) = cx+dy+k = 0$

Fixing y, imagine you are at an x', $x' > x$.

(or x'', $x'' < x$)

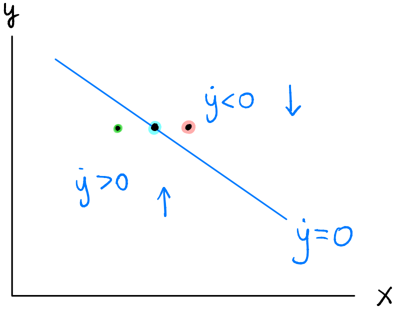
your new point is (x', y)

(or (x'', y))

Note $\dot{y}(t) = cx' + dy + k < 0$

$\dot{y}(t) = cx'' + dy + k > 0$

(because $c < 0$)



Assume $a, c, d < 0$, $b > 0$.

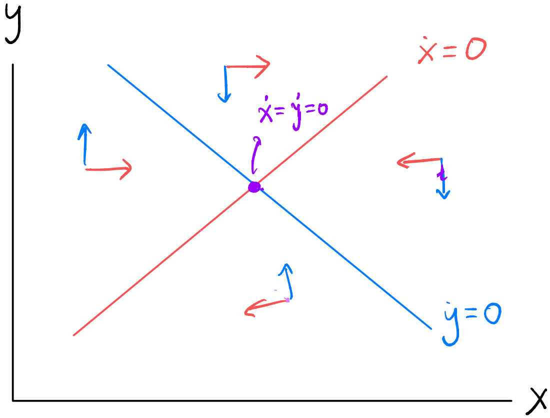
Building a Phase Diagram: System Dynamics

3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.

$\dot{x}$ determines all $\rightarrow, \leftarrow$ arrows

$\dot{y}$ determines all $\uparrow, \downarrow$ arrows

always!!



★ you could have more than one loci for each $\dot{x}_i=0$
but the same idea holds, each loci separates the plane in two regions w.r. to the movement of the variable! ☺

Example: Neoclassical Growth Model

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\eta} - 1}{1-\eta}\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t),$$

$$k(0) = k_0 > 0,$$

$$k(t) \geq 0 \qquad \lim_{t\to\infty} k(t) \geq 0 \Rightarrow \text{TVC}$$

$$\tilde{\mathcal{H}}(t) = \frac{c(t)^{1-\eta} - 1}{1-\eta} + \mu(t)\left[Af(k(t)) - \delta k(t) - c(t)\right]$$

Conditions for an optimum

① $\tilde{H}_c = 0$ $\quad (c) \quad c(t)^{-\eta} - \mu(t) = 0$ ✓

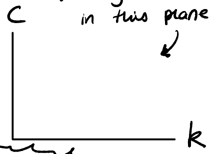
② $-\tilde{H}_k = \dot{\mu}(t) - \rho\mu(t)$ $\quad (k) \quad \dot{\mu}(t) - \rho\mu(t) = -\mu(t)\left[Af'(k(t)) - \delta\right]$ ✓

③ $\tilde{H}_\mu = \dot{k}(t)$ $\quad (\mu) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$ ✓

④ TVC $\quad TC \quad \lim_{T\to\infty} e^{-\rho T}\mu(T)k(T) = 0$ $\qquad \mu(T) \geq 0$

Example: Neoclassical Growth Model

Note: our goal is to graph the phase diagram of $\dot{c}$ and $\dot{k}$ in this plane



From

$$\left\{\begin{array}{ll} (c) & c(t)^{-\eta} - \mu(t) = 0 \\ (k) & \dot{\mu}(t) - \rho\mu(t) = -\mu(t)\left[Af'(k(t)) - \delta\right] \quad : \mu(t) \end{array}\right.$$

$$(\mu) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$

See procedure in following slide

We can arrive to an ODE system in $c(t)$ and $k(t)$:

$$(\dot{k}) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$

$$(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\left(Af'(k(t)) - \rho - \delta\right)$$

With these 2 equations we will follow the same procedure to characterize the Phase Diagram.

From (c), (k) to (1)

(c) $\quad c(t)^{-\eta} - \mu(t) = 0 \rightarrow \mu(t) = c(t)^{-\eta}$

(k) $\quad \dot{\mu}(t) - \rho\mu(t) = -\mu(t)\left[Af'(k(t)) - \delta\right]$ $\quad / \div \mu(t) \rightarrow \quad \dfrac{\dot{\mu}(t)}{\mu(t)} = \rho - (Af'(k(t)) - \delta)$

$$= \rho + \delta - Af'(k(t))$$

we learned $\dfrac{\dot{\mu}(t)}{\mu(t)} = \rho + \delta - Af'(k(t))$ (★)

Now take $\frac{d}{dt}$ on both sides of (c) : $\mu(t) = c(t)^{-\eta}$

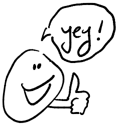
$$\frac{d}{dt}\mu(t) = \frac{d}{dt}c(t)^{-\eta}$$

$$\dot{\mu}(t) = -\eta\, c(t)^{-\eta-1}\,\dot{c}(t) \qquad / \text{ Divide by } \mu(t) = c(t)^{-\eta}$$

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{-\eta\, c(t)^{-\eta-1}\,\dot{c}(t)}{c(t)^{-\eta}}$$

$$= -\eta\,\frac{\dot{c}(t)}{c(t)}$$

Then : (1) $\quad \dfrac{\dot{c}(t)}{c(t)} = \dfrac{1}{\eta}\left(Af'(k(t)) - \rho - \delta\right)$



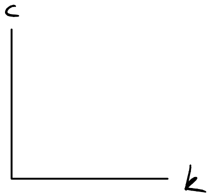
Phase Diagram in Neoclassical Model

$\dot{x}(t) = ax + by + h$

$\dot{y}(t) = cx + dy + k$

$$(\dot{k}) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$

$$(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\Big(Af'(k(t)) - \rho - \delta\Big)$$



1. **Loci**: graph the points (k, c) at which (1) $\dot{k}(t) = 0$ and (2) $\dot{c}(t) = 0$.

$$\begin{aligned} \dot{k}(t) = 0 \quad &\Longleftrightarrow \quad c(t) = Af(k(t)) - \delta k(t) \\ \dot{c}(t) = 0 \quad &\Longleftrightarrow \quad Af'(k(t)) = \rho + \delta \end{aligned}$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

Neoclassical Model Phase Diagram: $\dot{k}(t) = 0$

$$(\dot{k}) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$

1. **Loci**: graph the points (k, c) at which (1) $\dot{k}(t) = 0$.

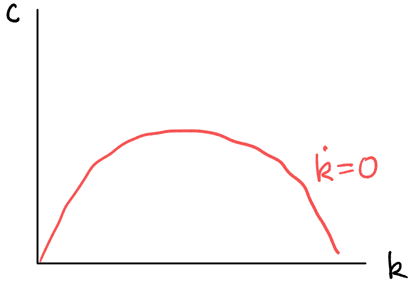
$$\dot{k}(t) = 0 \quad \Longleftrightarrow \quad c(t) = Af(k(t)) - \delta k(t)$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

Step 1:
just graph
$\dot{k}(t) = 0$

$c(t) = Af(k(t)) - \delta k(t)$

is concave and
begins at the origin
(if $k=0$, $c=0$)



Neoclassical Model Phase Diagram: $\dot{k}(t) = 0$

$$(\dot{k}) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$

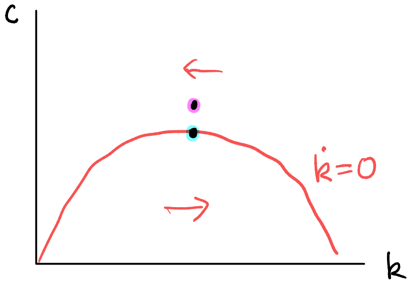
1. **Loci**: graph the points (k, c) at which (1) $\dot{k}(t) = 0$.

$$\dot{k}(t) = 0 \quad \Longleftrightarrow \quad c(t) = Af(k(t)) - \delta k(t)$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

$\rightarrow$ Hint: Take (k, c) as given, then move c and see what happens to $\dot{k}$.

Step 2:
From $\dot{k}=0$, graph winds.
($\leftarrow$ $\dot{k}<0$; $\rightarrow$ $\dot{k}>0$)



(k, c) $\dot{k}(t) = 0$

(k, c') $\dot{k}(t) < 0$

$c' > c$

Building a Phase Diagram: $\dot{c}(t) = 0$

$$(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\left(Af'(k(t)) - \rho - \delta\right)$$

1. **Loci**: graph the points (k, c) at which (2) $\dot{c}(t) = 0$.

$$\dot{c}(t) = 0 \quad \Longleftrightarrow \quad Af'(k(t)) = \rho + \delta$$

$f'(k(t)) = \frac{\rho+\delta}{A}$

$k(t) = (f')^{-1}\left[\frac{\rho+\delta}{A}\right]$

$k^{ss} =$

or k^* ☺

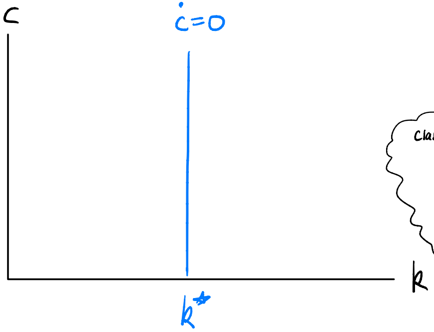
2. **Winds**: Characterize the behavior of the system in each area divided by (2).

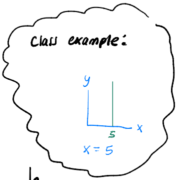
Step 1

just graph $\dot{c}=0$

Note this line only depends on k, not on c.

there is one value for k at which $\dot{c}=0$, $k = k^{ss}$





Building a Phase Diagram: $\dot{c}(t) = 0$

$$(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\Big(Af'(k(t)) - \rho - \delta\Big)$$

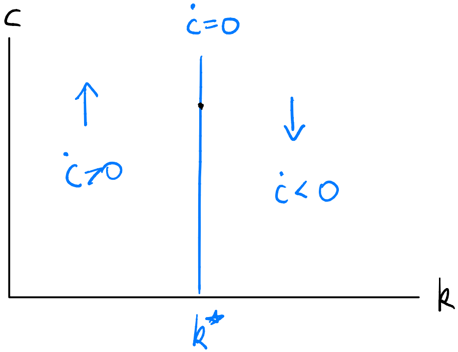
k^{ss}

1. **Loci**: graph the points (k, c) at which (2) $\dot{c}(t) = 0$.

$$\dot{c}(t) = 0 \iff Af'(k(t)) = \rho + \delta \rightarrow k^* = f'^{-1}\left(\frac{\rho+\delta}{A}\right)$$

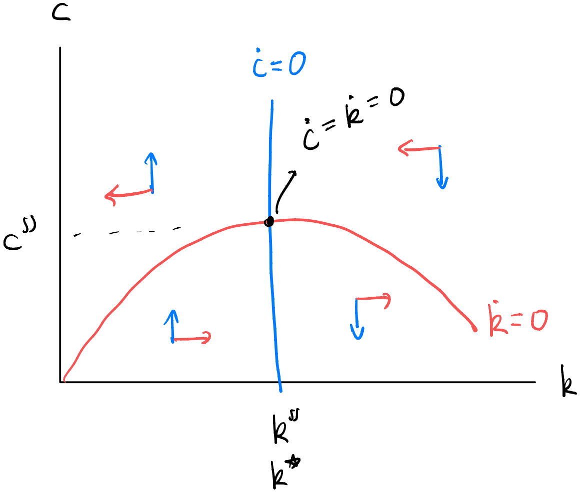
2. **Winds**: Characterize the behavior of the system in each area divided by (2).

$\rightarrow$ Hint: Take (k, c) as given, then move k and see what happens to $\dot{c}$.



Building a Phase Diagram: System Dynamics

3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.



Plan for Today

Phase Diagrams: Foundations

Example: Neoclassical Growth Model

Practice Question: PS1 D. Natural Resources

PS1. D. Capital Accumulation and Natural Resources

Consider an economy with an aggregate output technology that depends on human-made physical capital, k, and energy from a natural resource, z. The technology is:

$$y(t) = k(t)^{\eta} z(t)^{1-\eta},$$

where $\eta \in (0, 1)$.

At each instant, capital changes according to the law of motion:

$$\dot{k}(t) = y(t) - \delta k(t) - c(t),$$

where $c(t)$ represents the flow of output consumed at t, $\delta > 0$, and $k(0) = k_0 > 0$.

The social planner aims to maximize the objective function:

$$U = \int_0^{\infty} c(t) e^{-\rho t} dt,$$

where $\rho > 0$, subject to:

$$c(t) \geq \underline{c}, \quad \forall t \geq 0,$$

and $\underline{c} \geq 0$ is a minimal subsistence consumption level.

Assume the natural resource is renewable, with a maximal flow $\bar{z} > 0$:

$$z(t) \leq \bar{z}, \quad \forall t \geq 0.$$

PS1. D. Hamiltonian and Optimality Conditions

(a) Hamiltonian and Lagrangian:

$$\tilde{\mathcal{H}} = c(t) + \mu(t)\left[k(t)^{\eta} z(t)^{1-\eta} - \delta k(t) - c(t)\right]$$

with additional constraints incorporated into the Lagrangian:

$$\tilde{\mathcal{L}} = c(t) + \mu(t)\left[k(t)^{\eta} z(t)^{1-\eta} - \delta k(t) - c(t)\right] + \theta_1(t)(c(t) - \underline{c}) + \theta_2(t)(\bar{z} - z(t)).$$

- *State variable:* $k(t)$
- *Control variables:* $c(t), z(t)$

(b) Optimality Conditions:

$$\begin{array}{rll}
\tilde{\mathcal{L}}_c = 0 \quad (c) & 1 - \mu(t) + \theta_1(t) = 0 \quad \Longrightarrow \quad \mu(t) = \theta_1(t) + 1 \\
\tilde{\mathcal{L}}_z = 0 \quad (z) & \mu(t)(1-\eta)k(t)^{\eta} z(t)^{-\eta} - \theta_2(t) = 0 \\
-\tilde{\mathcal{L}}_k = \dot{\mu}(t) - \rho\mu(t) \quad (k) & \dot{\mu}(t) - \rho\mu(t) = -\tilde{\mathcal{L}}_k = \mu(t)\left[\delta - \eta k(t)^{\eta-1} z(t)^{1-\eta}\right] \\
\tilde{\mathcal{L}}_\mu = \dot{k}(t) \quad (\mu) & \dot{k}(t) = k(t)^{\eta} z(t)^{1-\eta} - \delta k(t) - c(t) \\
(\theta_1) & \theta_1(t)(c(t) - \underline{c}) = 0 \\
(\theta_2) & \theta_2(t)(\bar{z} - z(t)) = 0
\end{array}$$

PS1. D. Characterize steady state

(summary of the question prior to the part about phase diagrams)

- Question (c) asks you to characterize a steady state for z, k, c, that is, points at which $\dot{z} = 0$, $\dot{k} = 0$, $\dot{c} = 0$
- From (c) you learn $\dot{z} = 0$ always, $z(t) = \bar{z}$
- And

$$\dot{k}(t) = 0 \iff c(t) = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t)$$
$$\dot{c}(t) = 0 \iff \eta k(t)^{\eta-1}\bar{z}^{1-\eta} = \rho + \delta$$

- Question (d) asks you to provide conditions on the co-state for $c(t) = \underline{c}$ and $c(t) = \bar{c}$ (ss) to be optimal
- From (d) we learn that

$$\text{while } \mu(t) > 1 \implies c(t) = \underline{c}$$
$$\text{when } \mu(t) = 1 \implies c(t) = \bar{c}$$

- Note also that $\dot{\mu}(t) = 0 \iff \dfrac{\dot{\mu}(t)}{\mu(t)} = \rho + \delta - \eta k(t)^{\eta-1}\bar{z}^{1-\eta}$
- So if $k(0) < \bar{k}$, $\dot{\mu} < 0$: agent chooses subsistence consumption level, accumulates capital and watches $\mu \to 1$

PS1. D. Phase Diagram on the Space (k, μ)

The behavior of the co-state variable $\mu(t)$ determines the following conditions:

$$\mu(t) \begin{cases} > 1 \quad \Longrightarrow \; c(t) = \underline{c}, \quad \dot{k} = k(t)^{\eta} \bar{z}^{1-\eta} - \delta k(t) - \underline{c} \\ \qquad \dot{k} = 0 \quad \Longleftrightarrow \quad k(t)^{\eta} \bar{z}^{1-\eta} = \delta k(t) + \underline{c} \\ \\ = 1 \quad \Longrightarrow \; c(t) = \bar{c} \\ \qquad \dot{k} = 0 \quad \Longleftrightarrow \quad k^{ss} = \bar{z} \left(\frac{\delta+\rho}{\eta} \right)^{1/(\eta-1)} \end{cases}$$

Great! Now follow the steps:

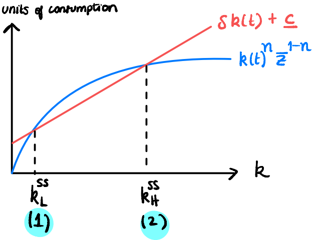
1. **Loci**: graph the points (k, μ) at which (1) $\dot{k}(t) = 0$ and (2) $\dot{\mu}(t) = 0$.
2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

PS1. D. (1) $\dot{k} = 0$

$$
\mu(t) \begin{cases} > 1 \implies c(t) = \underline{c}, \quad \dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} \\ \quad \dot{k} = 0 \iff k(t)^{\eta}\bar{z}^{1-\eta} = \delta k(t) + \underline{c} \\ = 1 \implies c(t) = \bar{c} \\ \quad \dot{k} = 0 \iff k^{ss} = \bar{z}\left(\frac{\delta+\rho}{\eta}\right)^{1/(\eta-1)} \end{cases}
$$

$\Rightarrow$ 2 levels of k for which $\dot{k}=0$ when $\mu(t)>1$

Graphically :



(3)

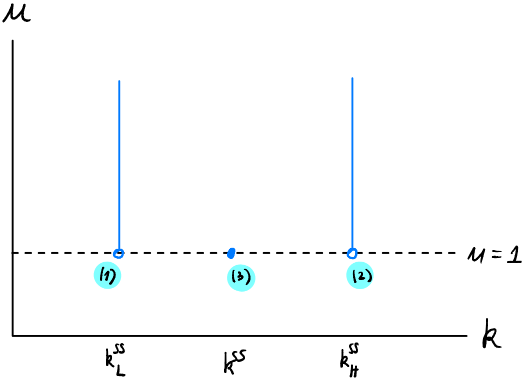
$\Rightarrow$ 1 level of k for which $\dot{k}=0$ when $\mu(t)=1$

1. **Loci**: graph the points (k, μ) at which (1) $\dot{k}(t) = 0$.
2. **Winds**: Characterize the behavior of the system in each area divided by (2).

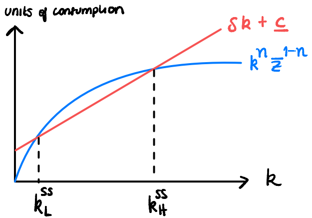
PS1. D. (1) $\dot{k} = 0$

1. **Loci**: graph the points (k, μ) at which (1) $\dot{k}(t) = 0$.
2. **Winds**: Characterize the behavior of the system in each area divided by (1).

Step 1 : Graph all the points (k, μ) at which $\dot{k} = 0$



(1) Step 2 : **Winds**: Characterize the behavior of the system in each area divided by (2).



First recall k_L^{ss} and k_H^{ss} are k for which $\dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} = 0$

Note that

$k < k_L^{ss}$: $\delta k + \underline{c} > k^{\eta}\bar{z}^{1-\eta} \Rightarrow \dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} < 0$

$k > k_L^{ss}$: $\delta k + \underline{c} < k^{\eta}\bar{z}^{1-\eta} \Rightarrow \dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} > 0$

$k < k_H^{ss}$: $\delta k + \underline{c} < k^{\eta}\bar{z}^{1-\eta} \Rightarrow \dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} > 0$

$k > k_H^{ss}$: $\delta k + \underline{c} > k^{\eta}\bar{z}^{1-\eta} \Rightarrow \dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} < 0$

(1) **Step 2:** Continuation

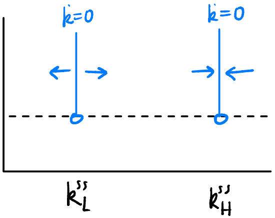
$k < k_L^{ss}$: $\delta k + \underline{c} > k^{\eta}\bar{z}^{1-\eta}$ $\Rightarrow$ $\dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} < 0$ $\leftarrow$

$k > k_L^{ss}$: $\delta k + \underline{c} < k^{\eta}\bar{z}^{1-\eta}$ $\Rightarrow$ $\dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} > 0$ $\rightarrow$

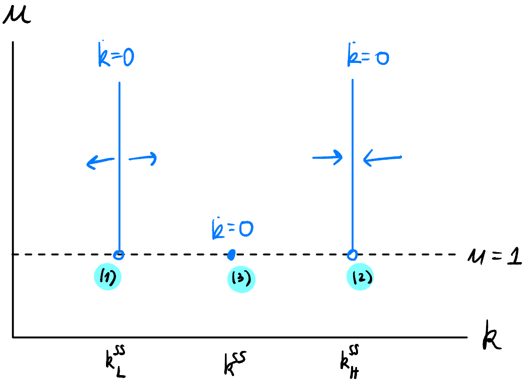
$k < k_H^{ss}$: $\delta k + \underline{c} < k^{\eta}\bar{z}^{1-\eta}$ $\Rightarrow$ $\dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} > 0$ $\rightarrow$

$k > k_H^{ss}$: $\delta k + \underline{c} > k^{\eta}\bar{z}^{1-\eta}$ $\Rightarrow$ $\dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} < 0$ $\leftarrow$

$\Longrightarrow$



(1) Step 2 : Continuation



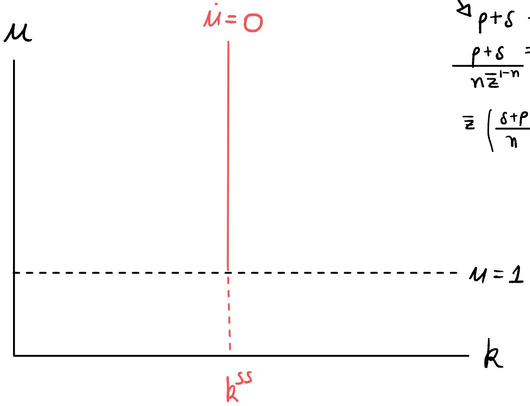
PS1. D. (2) $\dot{\mu} = 0$

1. **Loci**: graph the points (k, μ) at which (2) $\dot{\mu}(t) = 0$.

$$\dot{\mu}(t) = 0 \iff \frac{\dot{\mu}(t)}{\mu(t)} = \rho + \delta - \eta k(t)^{\eta-1} \bar{z}^{1-\eta} = 0$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2).

Step 1:
Graph
$\dot{\mu} = 0$



$$\rho + \delta = \eta k(t)^{\eta-1} \bar{z}^{1-\eta}$$

$$\frac{\rho+\delta}{\eta \bar{z}^{1-\eta}} = k(t)^{\eta-1} \quad /(\ )^{\frac{1}{\eta-1}}$$

$$\bar{z}\left(\frac{\delta+\rho}{\eta}\right)^{\frac{1}{\eta-1}} = k(t)$$

$$= k^{ss}$$

(2) Step 2 : **Winds**: Characterize the behavior of the system in each area divided by (2).

$$\frac{\dot{\mu}(t)}{\mu(t)} = \rho + \delta - \eta k(t)^{\eta-1}\bar{z}^{1-\eta}$$

$\dot{\mu}(t) = 0 \iff k^{ss}$

$k < k^{ss} \Rightarrow k^{\eta-1} > (k^{ss})^{\eta-1}$

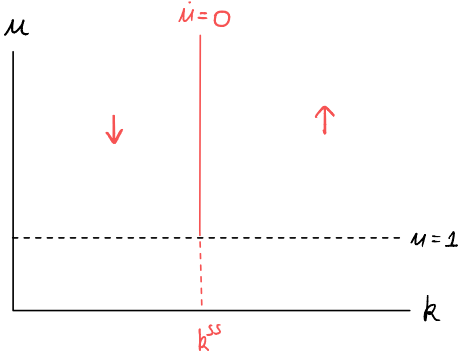
$\eta k^{\eta-1}\bar{z}^{1-\eta} > \eta (k^{ss})^{\eta-1}\bar{z}^{1-\eta}$

$\dot{\mu}(t) < 0$ ↓

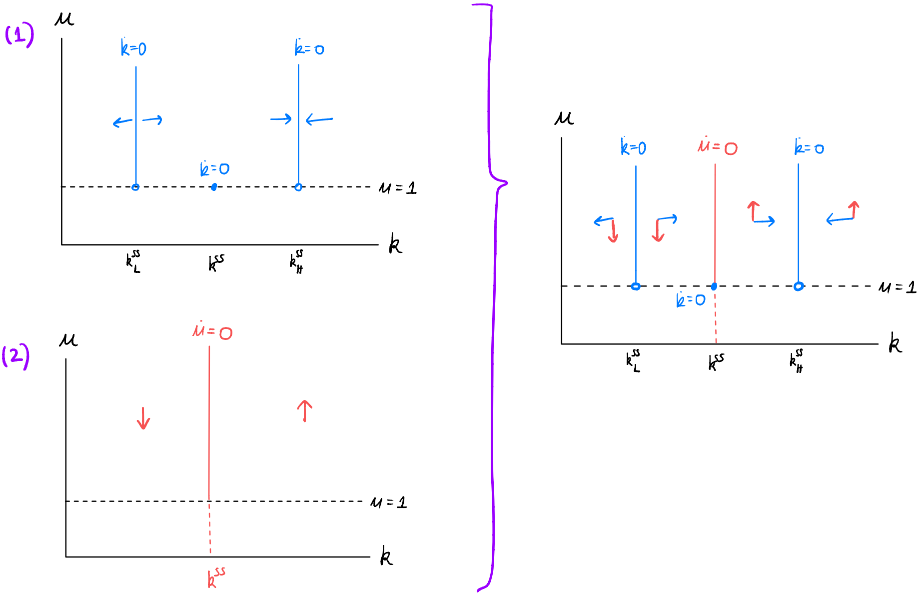
$k > k^{ss} \Rightarrow k^{\eta-1} < (k^{ss})^{\eta-1}$

$\eta k^{\eta-1}\bar{z}^{1-\eta} < \eta (k^{ss})^{\eta-1}\bar{z}^{1-\eta}$

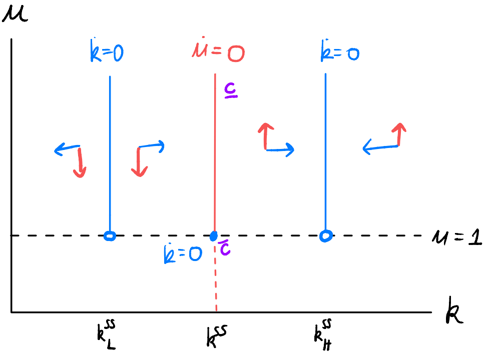
$\dot{\mu}(t) > 0$ ↑



PS1. D. (1) + (2) + Winds = Phase Diagram



Extra consumption



$\bar{c}$: consumption in ss

$\underline{c}$: Subsistence level consumption

Extra **A possible manifold or solution**

Note that if $k_0 < k^{ss}$, $\mu(t) > 1 \Rightarrow c(t) = \underline{c}$

until $\mu(t) \to 1$ & $c(t) = \bar{c}$ ($k(t) = k^{ss}$)

Graphically: