Econ 210B
Discussion Session N°3

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Plan for Today

Phase Diagrams: Foundations

Example: Neoclassical Growth Model

Practice Question: PS1 D. Natural Resources

Some Definitions

- **System of Ordinary Differential Equations (ODE)**: Set of equations involving multiple interdependent variables and their rate of change to a single independent variable (typically t). Example:

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), \ldots, x_n(t), t) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), \ldots, x_n(t), t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_1(t), x_2(t), \ldots, x_n(t), t)
\end{aligned}$$

  Where $\dot{x}_i(t) \equiv \frac{dx_i(t)}{dt}$ for each $i = 1, 2, \ldots, n$.

- **Phase Diagram**: Used to visualize the behavior of an ODE system (typically with 2 variables). Includes:
  - **Loci**: Points were one variable is at rest, i.e. $\dot{x}_i(t) = 0$.
  - **Regions**: Each locus divides the graph in 2 regions with respect to the sign of $\dot{x}_i(t)$.
  - **Winds**: To illustrate change of the variable in x-axis (y-axis) we use $\rightarrow$ $\leftarrow$ $(\downarrow \uparrow)$

Building a Phase Diagram

Think about a general 2 variable system (y and x) defined by the ODE:

$$\dot{x}(t) = ax + by + h$$
$$\dot{y}(t) = cx + dy + k$$

1. **Loci**: graph the points (x, y) at which (1) $\dot{x}(t) = 0$ and (2) $\dot{y}(t) = 0$.

$$\dot{x}(t) = 0 \quad \Longleftrightarrow ax + by + h = 0 \Longleftrightarrow y = -\frac{a}{b}x - h$$
$$\dot{y}(t) = 0 \quad \Longleftrightarrow cx + dy + k = 0 \Longleftrightarrow y = -\frac{c}{d}x - k$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

Building a Phase Diagram: (1) $\dot{x}(t) = 0$

1. **Loci**: graph the points (x, y) at which (1) $\dot{x}(t) = 0$.

$$\dot{x}(t) = 0 \iff ax + by + h = 0 \iff y = -\frac{a}{b}x - h$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1).

Assume $a, c, d < 0,\ b > 0$.

Building a Phase Diagram: (1) $\dot{x}(t) = 0$

1. **Loci**: graph the points (x, y) at which (1) $\dot{x}(t) = 0$.

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2. **Winds**: Characterize the behavior of the system in each area divided by (1).

$\rightarrow$ Hint: Take (x, y) as given, then move y and see what happens to $\dot{x}$.

Assume $a, c, d < 0,\ b > 0$.

Building a Phase Diagram: (2) $\dot{y}(t) = 0$

1. **Loci**: graph the points (x, y) at which (2) $\dot{y}(t) = 0$.

$$\dot{y}(t) = 0 \iff cx + dy + k = 0 \iff y = -\frac{c}{d}x - k$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2).

Assume $a, c, d < 0,\ b > 0$.

Building a Phase Diagram: (2) $\dot{y}(t) = 0$

1. **Loci**: graph the points (x, y) at which (2) $\dot{y}(t) = 0$.

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$\rightarrow$ Hint: Take (x, y) as given, then move x and see what happens to $\dot{y}$.

Assume $a, c, d < 0,\ b > 0$.

Building a Phase Diagram: System Dynamics

3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.

Example: Neoclassical Growth Model

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\eta}-1}{1-\eta}\, dt$$
$$\text{s.t.} \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t),$$
$$k(0) = k_0 > 0,$$
$$k(t) \geq 0$$

$$\tilde{\mathcal{H}}(t) = \frac{c(t)^{1-\eta}-1}{1-\eta} + \mu(t)\big[Af(k(t)) - \delta k(t) - c(t)\big]$$

Conditions for an optimum

$$(c) \quad c(t)^{-\eta} - \mu(t) = 0$$
$$(k) \quad \dot{\mu}(t) - \rho\mu(t) = -\mu(t)\big[Af'(k(t)) - \delta\big]$$
$$(\mu) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$
$$TC \quad \lim_{T\to\infty} e^{-\rho T}\mu(T)k(T) = 0$$

Example: Neoclassical Growth Model

From

$$
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\end{aligned}
$$

We can arrive to an ODE system in $c(t)$ and $k(t)$:

$$
\begin{aligned}
&(\dot{k}) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t) \\
&(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\Big(Af'(k(t)) - \rho - \delta\Big)
\end{aligned}
$$

With these 2 equations we will follow the same procedure to characterize the Phase Diagram.

Phase Diagram in Neoclassical Model

$$(\dot{k}) \quad \dot{k}(t) = Af(k(t)) - \delta k(t) - c(t)$$

$$(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\Big(Af'(k(t)) - \rho - \delta\Big)$$

1. **Loci**: graph the points (k, c) at which (1) $\dot{k}(t) = 0$ and (2) $\dot{c}(t) = 0$.

$$\begin{aligned} \dot{k}(t) = 0 \quad &\Longleftrightarrow \quad c(t) = Af(k(t)) - \delta k(t) \\ \dot{c}(t) = 0 \quad &\Longleftrightarrow \quad Af'(k(t)) = \rho + \delta \end{aligned}$$

2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
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Neoclassical Model Phase Diagram: $\dot{k}(t) = 0$

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2. **Winds**: Characterize the behavior of the system in each area divided by (1).

$\rightarrow$ Hint: Take (k, c) as given, then move c and see what happens to $\dot{k}$.

Building a Phase Diagram: $\dot{c}(t) = 0$

$$(\dot{c}) \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta}\Big(Af'(k(t)) - \rho - \delta\Big)$$

1. **Loci**: graph the points (k, c) at which (2) $\dot{c}(t) = 0$.

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Building a Phase Diagram: $\dot{c}(t) = 0$

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Building a Phase Diagram: System Dynamics

3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.

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Practice Question: PS1 D. Natural Resources

PS1. D. Capital Accumulation and Natural Resources

Consider an economy with an aggregate output technology that depends on human-made physical capital, k, and energy from a natural resource, z. The technology is:

$$y(t) = k(t)^{\eta} z(t)^{1-\eta},$$

where $\eta \in (0, 1)$.

At each instant, capital changes according to the law of motion:

$$\dot{k}(t) = y(t) - \delta k(t) - c(t),$$

where $c(t)$ represents the flow of output consumed at t, $\delta > 0$, and $k(0) = k_0 > 0$.

The social planner aims to maximize the objective function:

$$U = \int_0^{\infty} c(t) e^{-\rho t} dt,$$

where $\rho > 0$, subject to:

$$c(t) \geq \underline{c}, \quad \forall t \geq 0,$$

and $\underline{c} \geq 0$ is a minimal subsistence consumption level.

Assume the natural resource is renewable, with a maximal flow $\bar{z} > 0$:

$$z(t) \leq \bar{z}, \quad \forall t \geq 0.$$

PS1. D. Hamiltonian and Optimality Conditions

(a) Hamiltonian and Lagrangian:

$$\tilde{\mathcal{H}} = c(t) + \mu(t)\left[k(t)^{\eta} z(t)^{1-\eta} - \delta k(t) - c(t)\right]$$

with additional constraints incorporated into the Lagrangian:

$$\tilde{\mathcal{L}} = c(t) + \mu(t)\left[k(t)^{\eta} z(t)^{1-\eta} - \delta k(t) - c(t)\right] + \theta_1(t)(c(t) - \underline{c}) + \theta_2(t)(\bar{z} - z(t)).$$

- *State variable:* $k(t)$
- *Control variables:* $c(t), z(t)$

(b) Optimality Conditions:

$$\begin{aligned}
&(c) \quad 1 - \mu(t) + \theta_1(t) = 0 \quad \Longrightarrow \mu(t) = \theta_1(t) + 1 \\
&(z) \quad \mu(t)(1-\eta)k(t)^{\eta} z(t)^{-\eta} - \theta_2(t) = 0 \\
&(k) \quad \dot{\mu}(t) - \rho\mu(t) = -\tilde{\mathcal{L}}_k = \mu(t)\left[\delta - \eta k(t)^{\eta-1} z(t)^{1-\eta}\right] \\
&(\mu) \quad \dot{k}(t) = k(t)^{\eta} z(t)^{1-\eta} - \delta k(t) - c(t) \\
&(\theta_1) \quad \theta_1(t)(c(t) - \underline{c}) = 0 \\
&(\theta_2) \quad \theta_2(t)(\bar{z} - z(t)) = 0
\end{aligned}$$

PS1. D. Characterize steady state

- Question (c) asks you to characterize a steady state for z, k, c, that is, points at which $\dot{z} = 0$, $\dot{k} = 0$, $\dot{c} = 0$

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$$\begin{aligned} \dot{k}(t) = 0 \quad &\Longleftrightarrow \quad c(t) = k(t)^{\eta} \bar{z}^{1-\eta} - \delta k(t) \\ \dot{c}(t) = 0 \quad &\Longleftrightarrow \quad \eta k(t)^{\eta - 1} \bar{z}^{1-\eta} = \rho + \delta \end{aligned}$$

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- From (d) we learn that

$$\begin{aligned} \text{while } \mu(t) > 1 \quad &\implies \quad c(t) = \underline{c} \\ \text{when } \mu(t) = 1 \quad &\implies \quad c(t) = \bar{c} \end{aligned}$$

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- Note also that $\dot{\mu}(t) = 0 \quad \Longleftrightarrow \quad \dfrac{\dot{\mu}(t)}{\mu(t)} = \rho + \delta - \eta k(t)^{\eta-1} \bar{z}^{1-\eta}$

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- Note also that $\dot{\mu}(t) = 0 \quad \iff \quad \dfrac{\dot{\mu}(t)}{\mu(t)} = \rho + \delta - \eta k(t)^{\eta-1}\bar{z}^{1-\eta}$
- So if $k(0) < \bar{k}$, $\dot{\mu} < 0$: agent chooses subsistence consumption level, accumulates capital and watches $\mu \to 1$

PS1. D. Phase Diagram on the Space (k, μ)

The behavior of the co-state variable $\mu(t)$ determines the following conditions:

$$\mu(t) \begin{cases} > 1 \quad \Longrightarrow \; c(t) = \underline{c}, \quad \dot{k} = k(t)^{\eta}\bar{z}^{1-\eta} - \delta k(t) - \underline{c} \\ \qquad \dot{k} = 0 \quad \Longleftrightarrow \quad k(t)^{\eta}\bar{z}^{1-\eta} = \delta k(t) + \underline{c} \\ \\ = 1 \quad \Longrightarrow \; c(t) = \bar{c} \\ \qquad \dot{k} = 0 \quad \Longleftrightarrow \quad k^{ss} = \bar{z}\left(\frac{\delta+\rho}{\eta}\right)^{1/(\eta-1)} \end{cases}$$

PS1. D. Phase Diagram on the Space (k, μ)

The behavior of the co-state variable $\mu(t)$ determines the following conditions:

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Great! Now follow the steps:

1. **Loci**: graph the points (k, μ) at which (1) $\dot{k}(t) = 0$ and (2) $\dot{\mu}(t) = 0$.
2. **Winds**: Characterize the behavior of the system in each area divided by (1) and (2) separately.
3. **System Dynamics**: Put (1) + (2) + Winds together and you have characterized the dynamics of the system.
4. **Stability**: System can be globally stable, unstable or saddle-path stable.

PS1. D. (1) $\dot{k} = 0$

$$\mu(t) \begin{cases} > 1 & \Longrightarrow \; c(t) = \underline{c}, \quad \dot{k} = k(t)^{\eta} \bar{z}^{1-\eta} - \delta k(t) - \underline{c} \\ & \quad \dot{k} = 0 \quad \Longleftrightarrow \quad k(t)^{\eta} \bar{z}^{1-\eta} = \delta k(t) + \underline{c} \\ & \\ = 1 & \Longrightarrow \; c(t) = \bar{c} \\ & \quad \dot{k} = 0 \quad \Longleftrightarrow \quad k^{ss} = \bar{z} \left(\frac{\delta + \rho}{\eta} \right)^{1/(\eta - 1)} \end{cases}$$

1. **Loci**: graph the points (k, μ) at which (1) $\dot{k}(t) = 0$.
2. **Winds**: Characterize the behavior of the system in each area divided by (2).

PS1. D. (1) $\dot{k} = 0$

1. **Loci**: graph the points (k, μ) at which (1) $\dot{k}(t) = 0$.
2. **Winds**: Characterize the behavior of the system in each area divided by (2).

PS1. D. (2) $\dot{\mu} = 0$

1. **Loci**: graph the points (k, μ) at which (2) $\dot{\mu}(t) = 0$.

$$\dot{\mu}(t) = 0 \quad \iff \quad \mu(t) = 1$$

2. **Winds**: Characterize the behavior of the system in each area divided by (2).

PS1. D. (1) + (2) + Winds = Phase Diagram