Econ 210B
Discussion Session N°2

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Plan for Today

Practice Question: Constrained Investment

Review: Present vs. Current Value Hamiltonian

Review: Transversality Conditions*

Practice Question: Constrained Investment

$K_{t+1} = I_t + (1-\delta)K_t$

- Consider a firm that is producing output using capital $K(t)$ at the instantaneous rate $f(K(t))$, where f is strictly increasing, strictly concave, twice continuously differentiable, and $f(0) = 0$. Define $F(K(t)) = f(K(t)) - \mu K(t)$
- When capital is used in production it depreciates at the instantaneous rate $\mu K(t)$, where $\mu > 0$.
- The firm can invest to accumulate more capital. Let $I(t)$ denote net investment, then $\dot{K}(t) = I(t)$.
- Net investment is constrained to be $-\mu K(t) \leq I(t) \leq \bar{I}$, where $\bar{I} > 0$.
- The instantaneous flow of profits for the firm is, $f(K(t)) - \mu K(t) - I(t)$, and there exists a safe asset with instantaneous return r in which the firm can alternatively invest its profits.
- The firm is endowed with initial capital $K(0) = K_0$, and it maximizes profits for $t \in [0, T]$.

Practice Question: Constrained Investment

0. Write down the optimization problem. Indicate which are the state and control variables. Which is the law of motion for the state?
1. Write the Hamiltonian and Lagrangian associated with the firm's maximization problem and derive the necessary conditions for an optimal solution.
2. Characterize the path of capital and its shadow price for times in which the constraint on investment is binding from below, for times when it is binding from above, and for times when it is not binding.
3. Using the cases analyzed in (2), go as far as you can characterizing the path of optimal capital accumulation for the firm for $t \in [0, T]$.

Quick Summary

$$\max \quad \int_0^T g(x(t), y(t), t)\, dt$$

$$\begin{aligned} \text{s.t.} \quad & \dot{x}(t) = h(x(t), y(t), t), \\ & x(0) = x_0, \\ & x(T) \text{ unrestricted.} \end{aligned}$$

Where

- $y(t)$: control variable
- $x(t)$: state variable
- $\lambda(t)$: co-state variable

Present Value Hamiltonian:

$$H = g(x(t), y(t), t) + \lambda(t) h(x(t), y(t), t)$$

Maximum Principle: Optimal solution $x(t), y(t)$ must satisfy

$$\begin{aligned} &(y) && H_y = 0 \\ &(x) && \dot{\lambda}(t) = -H_x \\ &(\lambda) && \dot{x}(t) = H_\lambda \end{aligned}$$

Augmented Hamiltonian

$$\max \quad \int_0^T g(x(t), y(t), t)\, dt$$

$$\begin{aligned} \text{s.t.} \quad & \dot{x}(t) = h(x(t), y(t), t), \\ & x(0) = x_0, \\ & x(T) \text{ unrestricted}, \\ & y(t) \leq B(x(t), t). \end{aligned}$$

Augmented Hamiltonian:

$$\mathcal{L} = g(x(t), y(t), t) + \lambda(t) h(x(t), y(t), t) + \theta(t)(B(x(t), t) - y(t))$$

Maximum Principle:

$$\begin{aligned} (y) \quad & \mathcal{L}_y && = 0 \\ (x) \quad & \dot{\lambda}(t) && = -\mathcal{L}_x \\ (\lambda) \quad & \dot{x}(t) && = \mathcal{L}_\lambda \end{aligned}$$

KKT Conditions: $\mathcal{L}_y = 0$ ☑ and complementary slackness

$$\begin{aligned} (\theta) \quad & \theta(t)(B(x(t), t) - y(t)) && = 0 \\ & \theta(t) && \geq 0 \end{aligned}$$

Practice Question: Constrained Investment

Now back to the question:

0. Write down the optimization problem. Indicate which are the state and control variables. What is the law of motion for the state? What are the constraints on the control?

$-\mu K(t) \leq I(t) \leq \bar{I}$

- control: $I(t)$
- state: $K(t)$

$$\max \int_0^T e^{-rt} \left[F(K(t)) - I(t)\right] dt$$

KKT

$g(x) \geq 0$

Law of motion for the state: $\dot{K}(t) = I(t)$

$(K(0) = K_0)$

Constraints on the control: $-\mu K(t) \leq I(t) \Rightarrow 0 \leq I(t) + \mu K(t)$

From $K(t) \geq 0$: (boundary:) $K(T) \geq 0$

$I(t) \leq \bar{I} \Rightarrow 0 \leq \bar{I} - I(t)$

(TVC) $\lambda(T) K(T) = 0$; $\lambda(T) \geq 0$

Practice Question: Constrained Investment

1. Write the Hamiltonian and Lagrangian associated with the firm's maximization problem and derive the necessary conditions for an optimal solution.

(P) $$\int_0^T e^{-rt}\left(F(K(t)) - I(t)\right) dt$$

s.t. $$\dot{K}(t) = I(t)$$

$$I(t) + \mu K(t) \geq 0$$

$$\bar{I} - I(t) \geq 0$$

$$H = e^{-rt}\left(F(K(t)) - I(t)\right) + \lambda(t)\left(I(t)\right)$$

$$\mathcal{L} = e^{-rt}\left(F(K(t)) - I(t)\right) + \lambda(t)\left(I(t)\right) + \theta_1(t)\left(I(t) + \mu K(t)\right) + \theta_2(t)\left[\bar{I} - I(t)\right]$$

Practice Question: Constrained Investment

1. Write the Hamiltonian and Lagrangian associated with the firm's maximization problem and derive the necessary conditions for an optimal solution.

$$\mathcal{L} = e^{-rt}(F(K(t)) - I(t)) + \lambda(t)(I(t)) + \theta_1(t)(I(t) + \mu K(t)) + \theta_2(t)[\bar{I} - I(t)]$$

Conditions for an optimum (Maximum Principle)

(I) $\mathcal{L}_I = 0 \rightarrow -e^{-rt} + \lambda(t) + \theta_1(t) - \theta_2(t) = 0$

(K) $\dot{\lambda}(t) = -\mathcal{L}_K \rightarrow \dot{\lambda}(t) = -[e^{-rt} F'(K(t)) + \theta_1(t)\mu]$

(λ) $\dot{K}(t) = I(t)$

TV condition on the co-state

(TV) $\lambda(T)K(T) = 0, \quad \lambda(T) \geq 0$

Complementary slackness

(θ_1) $\theta_1(t)(I(t) + \mu K(t)), \quad \theta_1(t) \geq 0$

(θ_2) $\theta_2(t)[\bar{I} - I(t)], \quad \theta_2(t) \geq 0$

Practice Question: Constrained Investment

2. Characterize the path of capital and its shadow price for times in which the constraint on investment is binding from below, for times when it is binding from above, and for times when it is not binding.

First we will characterize the path for $\lambda(t)$ using (I), (θ_1), (θ_2)

Practice Question: Constrained Investment

2. Characterize the path of capital and its shadow price for times in which the constraint on investment is binding from below, for times when it is binding from above, and for times when it is not binding.

   First we will characterize the path for $\lambda(t)$ using (I), (θ_1), (θ_2)

$\longrightarrow$ Recall constraint on investment: $-\mu K(t) \leq I(t) \leq \bar{I}$

Practice Question: Constrained Investment

2. Characterize the path of capital and its shadow price for times in which the constraint on investment is binding from below, for times when it is binding from above, and for times when it is not binding. $\lambda(t) = e^{-rt} - \theta_1(t) + \theta_2(t)$

First we will characterize the path for $\lambda(t)$ using (I), (θ_1), (θ_2)

$\longrightarrow$ Recall constraint on investment: $\underbrace{-\mu K(t) \leq I(t)}_{① \ \theta_1} \leq \bar{I}$ ② θ_2

Case (i): $I(t) = \bar{I}$

- I don't learn anything about $\theta_2(t) \Rightarrow \theta_2(t)[\bar{I} - I(t)]$
- I do learn about $\theta_1(t) = 0$

$\lambda(t) = e^{-rt} + \theta_2(t)$

Case (ii): $I(t) \in (-\mu K(t), \bar{I})$ $\theta_1(t) = \theta_2(t) = 0$ $\lambda(t) = e^{-rt}$

Case (iii): $I(t) = -\mu K(t)$ $\theta_2(t) = 0$ $\lambda(t) = e^{-rt} - \theta_1(t)$

Practice Question: Constrained Investment

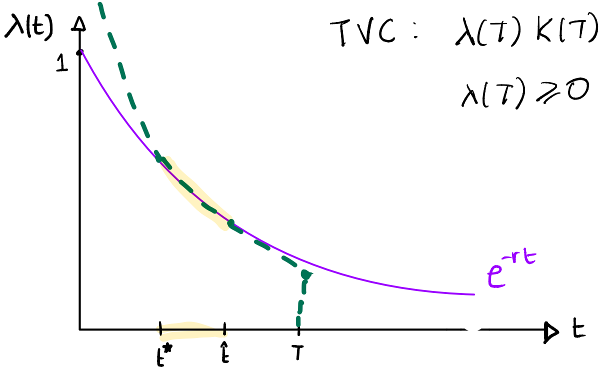
2. Possible path for $\lambda(t)$

$$t < t^* : \lambda(t) = e^{-rt} + \underbrace{\theta_2}(t)$$

$$t^* < t < \hat{t} : \lambda(t) = e^{-rt}$$

$$\underbrace{\hat{t} < t < T} : \lambda(t) = e^{-rt} - \theta_1(t)$$

Graphical representation:



TVC : $\lambda(T)\,K(T) = 0$

$\lambda(T) \geq 0$

Practice Question: Constrained Investment

3. Using the cases analyzed in (2), go as far as you can characterizing the path of optimal capital accumulation for the firm for $t \in [0, T]$.

Practice Question: Constrained Investment

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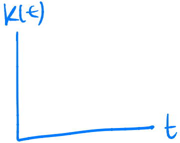
$\longrightarrow$ Strategy: characterize $K(t)$ consistent with the previous path for $\lambda(t)$, the optimality and slackness conditions

Practice Question: Constrained Investment

3. Using the cases analyzed in (2), go as far as you can characterizing the path of optimal capital accumulation for the firm for $t \in [0, T]$.

$\longrightarrow$ Strategy: characterize $K(t)$ consistent with the previous path for $\lambda(t)$, the optimality and slackness conditions



$$\underline{t < t^*} : \lambda(t) = e^{-rt} + \theta_2(t)$$

$$✓ \; I(t) = \bar{I}$$

$$✓ \; K(t) = K_0 + \bar{I}t$$

$$t^* < t < \hat{t} : \lambda(t) = e^{-rt}$$

$$I(t) \in (\mu K(t), \bar{I})$$

$$\hat{t} < t < T : \lambda(t) = e^{-rt} - \theta_1(t)$$

$$I(t) = -\mu \underline{K(t)}$$

Practice Question: Optimal Path for Capital

(K) $\dot{\lambda}(t) = -[e^{-rt} F'(K(t)) - \theta_1(t)]$ (2)

Strategy: $K(t)$ consistent with path for $\lambda(t)$, optimality and slackness conditions

$0 < t < t^*$: $\lambda(t) = e^{-rt} + \theta_2(t)$ (I)

$I(t) = \bar{I}$

$K(t) = K_0 + \bar{I}t$

$t^* < t < \hat{t}$: $\lambda(t) = e^{-rt}$ (I), (K):

$I(t) \in (\mu K(t), \bar{I})$

$\frac{\partial}{\partial t}$

$\dot{\lambda}(t) = -re^{-rt}$ (1)

$-\cancel{e^{-rt}} F'(K(t)) = -r\cancel{e^{-rt}}$

$F'(K(t)) = r$

$K(t) = K^{ss}$

$K(t) = K^{ss}$

$I(t) = 0$

$\hat{t} < t < T$: $\lambda(t) = e^{-rt} - \theta_1(t)$ (I)

$I(t) = -\mu K(t)$

$\dot{K}(t) = -\mu K(t)$

$K(t) = e^{-\mu t} K_0$

$I(t) = -\mu K(t)$

Note:
$$\frac{\partial e^{\mu t} K(t)}{\partial t} = \mu e^{\mu t} K(t) + e^{\mu t} \dot{K}(t)$$
$$= e^{\mu t} (\mu K(t) + \dot{K}(t))$$

From $\dot{K}(t) = -\mu K(t)$, we want to solve for $K(t)$:

$$\dot{K}(t) + \mu K(t) = 0 \quad / \cdot e^{\mu t}$$

$$e^{\mu t} (\dot{K}(t) + \mu K(t)) = 0$$

$$\frac{\partial e^{\mu t} K(t)}{\partial t} = 0 \quad / \int \quad dt$$

$$\int \frac{\partial e^{\mu t} K(t)}{\partial t} dt = \int 0 \, dt$$

By fundamental theorem of calculus

$$e^{\mu t} K(t) + A = B$$

$$e^{\mu t} K(t) = C \quad \forall t$$

$\Rightarrow$ Then it also holds for $t=0$:

$$e^{\mu 0} K(0) = C$$
$$1 \cdot K_0 = C$$

We conclude:
$$e^{\mu t} K(t) = K_0$$
$$K(t) = e^{-\mu t} K_0$$

Practice Question: Optimal Path for Capital

Investment

$$t < t^* : I(t) = \bar{I}$$

$$t^* < t < \hat{t} : I(t) = 0$$

$$\hat{t} < t < T : I(t) = -\mu K(t)$$

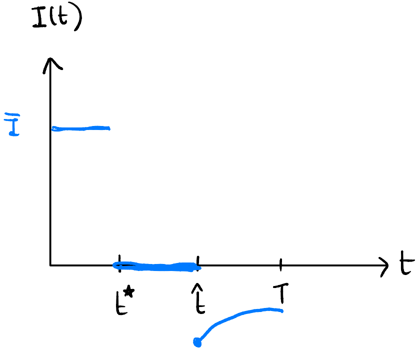
Capital

slope

$$t < t^* : K(t) = K(0) + \bar{I}t$$

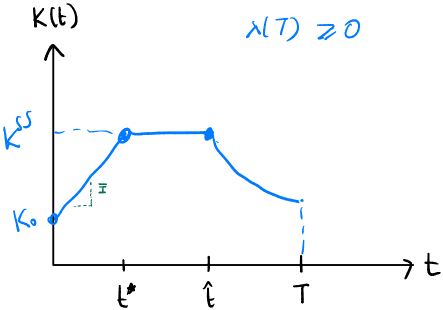
$$t^* < t < \hat{t} : K(t) = K^{ss}$$

$$\hat{t} < t < T : K(t) = e^{-\mu t'} K(0)$$



$\lambda(T)\,K(T) = 0$

$\lambda(T) \geq 0$



Present vs. Current Value Hamiltonian

$$\max \quad \int_0^T g(x(t), y(t), t)\, dt$$

$$\begin{aligned} \text{s.t.} \quad & \dot{x}(t) = h(x(t), y(t), t), \\ & x(0) = x_0, \\ & x(T) \text{ unrestricted.} \end{aligned}$$

If $g(x(t), y(t), t) = \bar{e}^{\rho t}\tilde{g}(x(t), y(t))$

Present Value Hamiltonian

$$H = g(x(t), y(t), t) + \lambda(t) h(x(t), y(t), t)$$

$\lambda(t) = e^{-\rho t} \mu(t)$

Maximum Principle:

(y)	$H_y = 0$
(x)	$\dot{\lambda}(t) = -H_x$
(λ)	$\dot{x}(t) = H_\lambda$

Current Value Hamiltonian

$$\tilde{H} = \tilde{g}(x(t), y(t), t) + \mu(t) h(x(t), y(t), t)$$

with $\mu(t) = e^{\rho t}\, \lambda(t)$

Maximum Principle:

(y)	$\tilde{H}_y = 0$
(x)	$\dot{\mu}(t) - \rho\mu(t) = -\tilde{H}_x$
(μ)	$\dot{x}(t) = \tilde{H}_\mu$

Review: Transversality Conditions

- **Boundary conditions** are assumptions about the behavior of a state variable, in a boundary situation

  $\longrightarrow$ for instance, at the final period T in finite horizon, or at $t \to \infty$ in infinite horizon

- **Transversality conditions** are conditions for an optimum, i.e. when we characterize the optimal path they must hold

- From the boundary condition for a state variable, we can derive transversality conditions for the related co-state variable
  - $\lambda(T)$ in finite horizon or $\lim_{t\to\infty} \lambda(t)$ in infinite horizon
  - $\mu(T)$ in finite horizon or $\lim_{t\to\infty} \mu(t)$ in infinite horizon

Boundary Condition on x and Transversality Condition on λ

Horizon	Boundary Condition	Implied Transversality Condition on Co-state
T	$x(T)$ unrestricted $x(T) = \bar{x}$ $x(T) \geq \bar{x}$	$\lambda(T) = 0$ $\lambda(T)$ unrestricted 1. $\lambda(T)(x(T) - \bar{x}) = 0$ 2. $\lambda(T) \geq 0$
∞	$\lim_{t\to\infty} x(t)$ unrestricted $\lim_{t\to\infty} x(t) = \bar{x}$ $\lim_{t\to\infty} x(t) \geq \bar{x}$	$\lim_{t\to\infty} \lambda(t) = 0$ $\lim_{t\to\infty} \lambda(t)$ unrestricted 1. $\lim_{t\to\infty} \lambda(t)\left[x(t) - \bar{x}\right] = 0$ 2. $\lim_{t\to\infty} \lambda(t) \geq 0$

Note:

From 1. and 2.: $x(T) > \bar{x} \Rightarrow \lambda(T) = 0$, $\quad x(T) = \bar{x} \Rightarrow \lambda(T) \geq 0$ ($\equiv$ with $t \to \infty$ case)

Replace $\lambda(t) = e^{-\rho t}\mu(t)$ and you've got your current value TVC.