Econ 210B
Discussion Session N°10

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Plan for Today

Review on Matching

PS4 Problem E: Matching Frictions and Labor-Augmenting Productivity

Review on Matching: Poisson Process

- Poisson Distribution: N is a Poisson random variable if it follows the Poisson probability distribution

$$Pr(N = n) = f_n(n) = \frac{\Lambda}{n!} e^{\Lambda}$$

$E[N] = V[N] = \Lambda$

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- Homogeneous Poisson Process: $\Lambda = \nu\lambda$
  - ν: measure
  - λ: rate or intensity, average numbers of points of the process per unit of length, mean rate or intensity

Matches arrive according to a Poisson Process

- Matches arrive according to a generalized Poisson process with rate

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- Note from the Poisson process properties:
  - $Pr(N_{[t,t+\Delta t]} = 1) \approx m_t \Delta t$
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- Less formally but more intuitive, we call m_t the matching function or number of new job matches per unit of time
- We assume it is increasing in both arguments, concave and hd1/CRS

$$\text{CRS:} \quad m(\alpha U, \alpha V) = \alpha m(U, V)$$
$$\implies m(\frac{U}{L}, \frac{V}{L}) = \frac{1}{L} m(U, V)$$
$$m(u, v) = \frac{1}{L} m(U, V)]$$

Review on Matching: Building Blocks

Matching function	$m(U, V)$	$m_i > 0,\ m_{ii} < 0,$ hd1/CRS
Tightness	$\theta = \frac{v}{u}$	$v = \frac{V}{L},\ u = \frac{U}{L}$
Job filling rate	$q(\theta) = \frac{m(U,V)}{V} = m\big(\frac{1}{\theta}, 1\big)$	decreasing in θ
Job finding rate	$\theta\, q(\theta) = \frac{m(U,V)}{U} = m\big(1, \theta\big)$	increasing in θ
Job destruction rate	λ	exogenous

Unemployment dynamics	$\dot{u} = \lambda(1-u) - \theta q(\theta) u$	in ss: $\dot{u} = 0$
Beveridge curve	BC: $u = \frac{\lambda}{\lambda + \theta q(\theta)}$	negative relationship btw u, v
		shifts $\nearrow$ with λ

Review on Matching: Value Functions

Firms: Post 1 vacancy, if filled generate p, if not pay cost pc.

Value of vacancy	$\mathcal{V}$	$r\mathcal{V} = -p\,c + q(\theta)\big(J - \mathcal{V}\big)$
Value of filling	J	$rJ = p - w + \lambda\big(\mathcal{V} - J\big)$

Workers: Risk neutral, max. PV of income.

Value of unemployment	$\mathcal{U}$	$r\mathcal{U} = z + \theta\, q(\theta)\,\big(W - \mathcal{U}\big)$
Value of employment	W	$rW = w + \lambda\,\big(\mathcal{U} - W\big)$

Review of Matching: Solving for the Equilibrium

Labor Demand

Replace $\mathcal{V} = 0$ in $\mathcal{V}$:

$$J = \frac{pc}{q(\theta)} \quad (\star)$$

Substituting into rJ:

$$\text{JC} \quad w = p - (r+\lambda)\frac{pc}{q(\theta)}$$

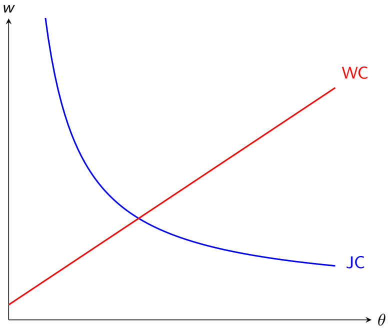
Wage Curve

Worker's wage solves the NB problem:

$$w_i = \arg\max\,(W_i - U)^{\beta}\,(J_i - \mathcal{V})^{1-\beta}$$
$$\text{s.t.} \quad S_i = W_i - \mathcal{U} + J_i - \mathcal{V} \quad (\star)$$

Taking FOC, and replacing $r\mathcal{U}$ from $\mathcal{U}$ using $\star$ and $\star$:

$$\text{WC} \quad w = (1-\beta)\,z + \beta\,p\,(1 + c\,\theta)$$



Extra: Derivation of WC

Part of the derivation of WC is in lecture 17, slides 22 to 23. Here I include the part that is missing. Try working on it on your own and check this if needed.

$$\begin{aligned} w_i = {} & \arg\max \left(W_i - \mathcal{U}\right)^{\beta} \left(J_i - \mathcal{V}\right)^{1-\beta} \\ \text{s.t.} \quad & S_i = W_i - \mathcal{U} + J_i - \mathcal{V} \quad (\star) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial w_i} &= \beta (W_i - \mathcal{U})^{\beta-1} \frac{\partial W_i}{\partial w_i} \left(S_i - W_i - \mathcal{U}\right)^{1-\beta} - (1-\beta)(W_i - \mathcal{U})^{\beta} \left(S_i - W_i - \mathcal{U}\right)^{-\beta} \frac{\partial W_i}{\partial w_i} = 0 \\ \beta (W_i - U)^{\beta-1} \frac{\partial W_i}{\partial w_i} \left(S_i - W_i - \mathcal{U}\right)^{1-\beta} &= (1-\beta)(W_i - \mathcal{U})^{\beta} \left(S_i - W_i - \mathcal{U}\right)^{-\beta} \frac{\partial W_i}{\partial w_i} \\ \beta (S_i - W_i - \mathcal{U}) &= (1-\beta)(W_i - \mathcal{U}) \\ \beta S_i - \beta (W_i - \mathcal{U}) &= (1-\beta)(W_i - \mathcal{U}) \\ \beta S_i &= W_i - \mathcal{U} \end{aligned}$$

Note

$$\begin{aligned} S_i &= W_i - \mathcal{U} + J_i \\ &= \beta S_i + J_i \\ S_i &= \frac{1}{1-\beta} J_i \end{aligned}$$

Extra: Derivation of WC

From

$$\beta S_i = W_i - \mathcal{U}$$

$$W_i = \mathcal{U} - \beta S_i \quad (1)$$

From W

$$\begin{aligned} rW_i &= w_i + \lambda(\mathcal{U} - W_i) \\ W_i(r+\lambda) &= w_i + \lambda\mathcal{U} \\ W_i &= \frac{w_i + \lambda\mathcal{U}}{r+\lambda} \quad (2) \end{aligned}$$

$$\begin{aligned} (1) = (2): \quad \mathcal{U} - \beta S_i &= \frac{w_i + \lambda\mathcal{U}}{r+\lambda} \\ r\mathcal{U} + \lambda\mathcal{U} - \beta(r+\lambda)S_i &= w_i + \lambda\mathcal{U} \\ r\mathcal{U} - \beta(r+\lambda)S_i &= w_i \end{aligned}$$

Replace $S_i = \frac{1}{1-\beta} J = \frac{1}{1-\beta} \frac{p-w_i}{r+\lambda}$ to arrive to

$$w_i = r\mathcal{U} + \beta(p - r\mathcal{U}) \quad (3)$$

Extra: Derivation of WC

Recall $J = \frac{pc}{q(\theta)}$ $\star$, and $W - \mathcal{U} = \frac{\beta}{1-\beta} J$ $\star$

From $\mathcal{U}$:

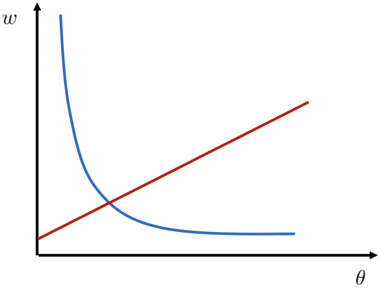
$$
\begin{aligned}
r\mathcal{U} \ &= \ z \ + \ \theta\, q(\theta) \left(W - \mathcal{U}\right) \\
&= z \ + \ \theta\, q(\theta) \left(\frac{\beta}{1-\beta} \frac{pc}{q(\theta)} \right) \\
&= z \ + \ \theta pc \left(\frac{\beta}{1-\beta} \right) \quad (4)
\end{aligned}
$$

Replace (4) into (3) and you arrive to the WC:

$$
\text{WC} \quad w = (1-\beta)\, z \ + \ \beta\, p \left(1 + c\,\theta\right)
$$

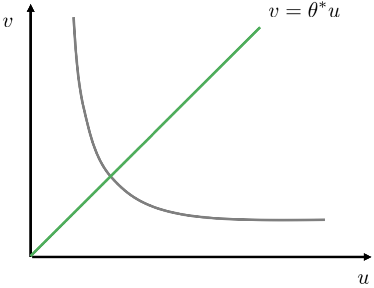
Review of Matching: Steady State Equilibrium

A Steady State Equilibrium is a triple (u, θ, w) that satisfies:



$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

$$w = (1 - \beta)z + \beta p(1 + c\theta)$$



$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

PS4 Problem E: Matching Frictions and Labor-Augmenting Productivity

Consider an economy with a representative firm that maximizes the present discounted value of profits:

$$\int_0^\infty e^{-rt}\left[AN(t) - w(t)N(t) - cAV(t)\right] dt.$$

- $N(t)$ denotes labor employed aby the firm, $V(t)$ the vacancies posted, $w(t)$ the wage, A the labor-augmenting productivity, $cA > 0$ the cost of posting a vacancy (where $c > 0$), and $r > 0$ the discount rate.
- In order to recruit workers, the firm posts vacancies. The firm generates $q(\theta(t))V(t)$ new hires each instant. The firm loses workers at the rate $\lambda N(t)$, with $\lambda > 0$.
- $V(t)$ determines labor market tightness $\theta(t)$, and $\theta(t) = \frac{v}{u}$, where v is the vacancy rate and u the unemployment rate. The function $q(\theta)$ is defined as $q(\theta) = \theta^{-\alpha}$, $\alpha \in (0, 1)$.

PS4 Problem E: Matching Frictions and Labor-Augmenting Productivity

(a) Write the law of motion for labor employed by the firm as implied by the hiring frictions faced by the firm.

(b) Obtain all the necessary conditions for an optimal solution to the firm's problem, including the appropriate transversality condition.

(c) Suppose that $\theta(t)$ and $w(t)$ are constant and consider the steady state solution where $\dot{N}(t) = 0$. Obtain the job creation equation in the steady state and explain in what sense it represents a labor-demand curve.

(d) Suppose in the steady state the wage equation resulting from the Nash-bargaining solution between the firm and the worker is: $w = (1 - \beta)z + \beta A(1 + c\theta)$. Using both equations and diagrams, show how to represent the equilibrium in the labor market.

(e) Assume that labor productivity evolves over time according to the process $A(t) = A_0 e^{gt}$. Under this assumption, what is the relationship between the growth rate in labor-augmenting productivity, g, and the unemployment rate u? What is the economic mechanism behind such a relationship?

PS4 E (a): Write the Law of Motion for Labor Employed by the Firm

Write the law of motion for labor employed by the firm as implied by the hiring frictions faced by the firm.

- Write the Hamiltonian (current net present value) associated with the firm's maximization problem.
- Indicate what the control and state variables are.

$$\max \quad \int e^{-rt} \left[AN(t) - w(t)N(t) - cAV(t) \right] dt$$

$$\text{s.t.} \quad \dot{N}(t) = q(\theta(t))V(t) - \lambda N(t)$$

$$\tilde{\mathcal{H}} = AN(t) - w(t)N(t) - cAV(t) + \mu(t)\left[q(\theta(t))\, V(t) - \lambda N(t) \right]$$

control : $V(t)$

state : $N(t)$

PS4 E (b): Necessary Conditions for an Optimal Solution

Obtain all the necessary conditions for an optimal solution to the firm's problem.

$$\tilde{\mathcal{H}} = AN(t) - w(t)N(t) - cAV(t) + \mu(t)[q(\theta(t))V(t) - \lambda N(t)]$$

control : $V(t)$

state : $N(t)$

(V) $\tilde{\mathcal{H}}_V = 0 \rightarrow -cA + \mu(t)q(\theta(t)) = 0 \rightarrow cA = \mu(t)q(\theta(t))$

(N) $-\tilde{\mathcal{H}}_N = \dot{\mu}(t) - r\mu(t) \rightarrow -[A - w(t) - \mu\lambda] = \dot{\mu}(t) - r\mu(t)$

$r\mu(t) = A - w(t) - \mu(t)\lambda + \dot{\mu}(t)$

(μ) $\tilde{\mathcal{H}}_\mu = \dot{N}(t) \rightarrow \dot{N}(t) = q(\theta(t))V(t) - \lambda N(t)$

PS4 E (b): Necessary Conditions for an Optimal Solution

Obtain all the necessary conditions for an optimal solution to the firm's problem.

- Explain in what sense the co-state variable represents a shadow price and provide a "no-arbitrage" interpretation for the law of motion of the co-state variable.

$$\mathcal{V}\left[\begin{array}{l} 0 = -cA + q(\theta)[J - 0] \\ 0 = -cA + \mu(t)\, q(\theta(t)) \end{array}\right.$$

$$\mu(t) = J$$

$$J\left[\begin{array}{l} r\mu(t) = A - w(t) - \mu(t)\lambda + \dot{\mu}(t) \\ rJ = A - w + \lambda(0 - J) + \dot{J} \end{array}\right.$$

Value of vacancy $\mathcal{V}$ $\quad r\mathcal{V} = -pc + q(\theta)\big(J - \mathcal{V}\big)$

Value of filling J $\quad rJ = p - w + \lambda\big(\mathcal{V} - J\big)$

PS4 E (c): Steady State Solution

Suppose that $\theta(t)$ and $w(t)$ are constant and consider the steady state solution where $\dot{N}(t) = 0$.

- Obtain the job creation (JC) equation in the steady state and explain in what sense it represents a labor-demand curve.

Labor Demand

Replace $\mathcal{V} = 0$ in $\mathcal{V}$:

$$J = \frac{pc}{q(\theta)} \quad (\star)$$

Substituting into rJ:

$$\boxed{\text{JC} \quad w = p - (r+\lambda)\frac{pc}{q(\theta)}}$$

$$0 = -cA + q(\theta)[J - 0]$$

$$cA = q(\theta) J \rightarrow J = \frac{cA}{q(\theta)} \quad (1)$$

$$rJ = A - w + \lambda(0 - J) + \underset{\substack{\parallel \\ 0}}{\dot{J}}$$

$$rJ = A - w - \lambda J$$

$$(r+\lambda)J = A - w$$

$$J = \frac{A-w}{r+\lambda} \quad (2)$$

$(1) = (2)$

$$\frac{cA}{q(\theta)} = \frac{A-w}{r+\lambda}$$

$$\frac{cA}{q(\theta)}(r+\lambda) = A - w \longrightarrow w = A - (r+\lambda)\frac{cA}{q(\theta)} \quad \text{JC}$$

PS4 E (c): Steady State Solution

Suppose that $\theta(t)$ and $w(t)$ are constant and consider the steady state solution where $\dot{N}(t) = 0$.

- Obtain the job creation (JC) equation in the steady state and explain in what sense it represents a labor-demand curve.
- Obtain an expression for the Beveridge curve for this economy in the steady state and explain what determines the slope and location of the curve.

Start from law of motion

$$\dot{N}(t) = q(\theta(t))V(t) - \lambda N(t)$$

$\dot{N}(t) = 0$

$$q(\theta) V(t) = \lambda N(t) \qquad / : L(t)$$

$$q(\theta) \frac{V(t)}{L(t)} = \lambda \frac{N(t)}{L(t)}$$

$$q(\theta) v = \lambda (1-u) \qquad / : u \qquad \text{recall } \theta = \frac{v}{u}$$

$$q(\theta)\, \theta = \frac{\lambda}{u} - \lambda$$

(c) continuation :

$$\frac{\lambda}{u} = q(\theta)\theta + \lambda$$

$$\frac{u}{\lambda} = \frac{1}{q(\theta)\theta + \lambda}$$

$$u = \frac{\lambda}{q(\theta)\theta + \lambda}$$

slope : θ ; $q(\theta)$ — negative relation btw u-v —

$\uparrow v$

location : λ — λ shifts BC to the right —

PS4 E (d): Wage Equation and Equilibrium Representation

Suppose in the steady state the wage equation (WC) resulting from the Nash-bargaining solution between the firm and the worker is:

$$w = (1 - \beta)z + \beta A(1 + c\theta),$$

- where $\beta \in (0, 1)$ represents the bargaining strength of the worker, and $z \geq 0$ is unemployment income.
- Using both equations and diagrams, show how to represent the equilibrium in the labor market.
- Suppose a permanent increase in labor productivity occurs. Show the impact on the equilibrium variables in steady state.

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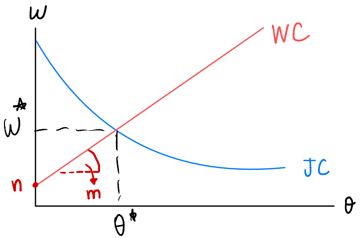
JC $\quad w = A - (\lambda + r)\, A c\, \theta^{\alpha}$

WC $\quad w = (1-\beta)z + \beta A(1 + c\theta)$

$$= \underbrace{(1-\beta)z + \beta A}_{n} + \underbrace{\beta A c}_{m}\, \theta$$



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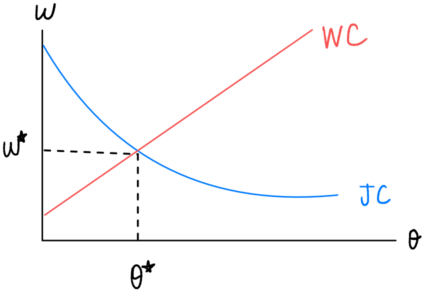
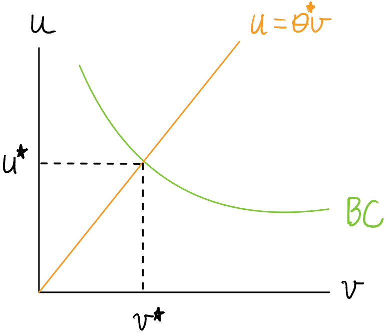




PS4 E (d): Wage Equation and Equilibrium Representation

- Suppose a permanent increase in labor productivity occurs. Show the impact on the equilibrium variables in steady state.

higher A

JC $\quad w = A - (\lambda + r) A c \theta^{\alpha} \quad (1)$

WC $\quad w = (1-\beta) z + \beta A (1 + c\theta) \quad (2)$

BC $\quad u = \dfrac{\lambda}{\lambda + \theta^{1-\alpha}} \quad \times$

$\theta^{*} = \dfrac{V}{u}$

(1) JC $\quad \dfrac{\partial w}{\partial A} = 1 - (\lambda + r) c \theta^{\alpha}$ is ≥ 0 if $1 \geq (\lambda + r) c \theta^{\alpha}$ $\quad$ JC $\nearrow$

(2) WC : n and m go up

PS4 E (d): Wage Equation and Equilibrium Representation

- Suppose a permanent increase in labor productivity occurs. Show the impact on the equilibrium variables in steady state.

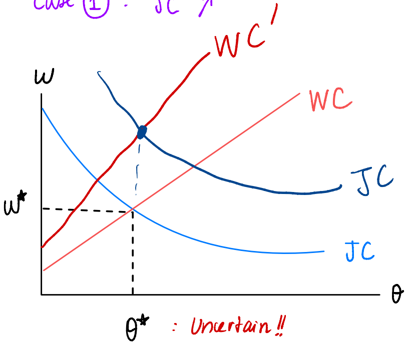
higher A

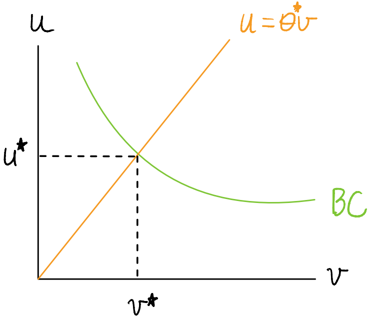
JC $\quad w = A - (\lambda + r) A c \theta^{\alpha}$: uncertain

WC $\quad w = (1-\beta) z + \beta A (1 + c\theta)$: n and m go up

BC $\quad u = \dfrac{\lambda}{\lambda + \theta^{1-\alpha}}$

case 1 : JC ↗





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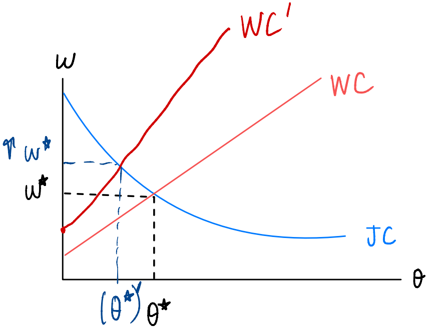
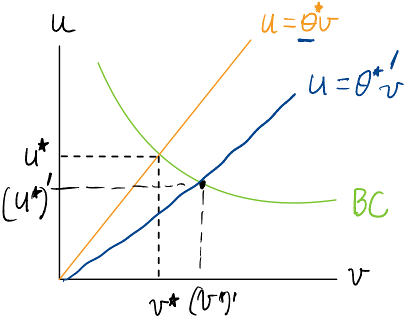
BC $\quad u = \dfrac{\lambda}{\lambda + \theta^{1-\alpha}}$

case 2 : JC doesn't change





PS4 E (e): Relationship between Growth Rate and Unemployment

Assume that labor productivity evolves over time according to the process:

$$A(t) = A_0 e^{gt},$$

- with $r > g > 0$ and $A_0 > 0$.
- Under this assumption, what is the relationship between the growth rate in labor-augmenting productivity, g, and the unemployment rate u?
- What is the economic mechanism behind such a relationship?