Econ 210B
Discussion Session N°1

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Plan for today

Introduction to 210B

Optimal Control vs Dynamic Programming

Exercise: Life-Cycle Consumption

210B and the Macro sequence

210 A

- ☑ **Methods**: Constrained optimization, Dynamic programming
- ☑ **Language**: Characterization of macro models, Neoclassical assumptions, BSP, Types of decentralized market equilibria
- ☑ **Topics**: Long term Growth, RBC, Ramsey Policies, Aiyagari

210 B

210 C

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- ☐ **All topics**: Financial frictions, Credit supply, Investment, Money!

More on 210B

- 4 Problem Sets
  - Follow the class closely
    1. Optimal Control
    2. Asset Pricing
    3. Aiyagari
    4. OLG, Search and Matching
  - Optional problems
  - We will work some in Discussions
  - All the study material you need!

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- General advice
  - Don't stress over grades, do worry about learning!
  - Remember the PhD is a long run, not a sprint. Take care of yourself and your team.

Optimal Control (210B) vs Dynamic Programming (210A)

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  - Continuous-Time framework
  - Necessary conditions lead to a system of differential equations that characterize the optimal trajectory and the mechanics of the system

Example: Neoclassical Model

Dynamic Programming

Sequential Problem:

$$\begin{aligned} \max_{\{k_{t+1}, c_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + k_{t+1} \leq F(k_t) + (1-\delta)k_t \end{aligned}$$

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Functional Equation Problem:

$$\begin{aligned} v(k) \quad &= \quad \max_{c, k'} \quad \{u(c) + \beta v(k')\} \\ \text{s.t.} \quad & \quad c + k' \leq F(k) + (1-\delta)k \end{aligned}$$

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Optimal Control

$$\max_{c(t),k(t)} \quad \int_0^{\infty} e^{-\rho t} u(c(t)) \, dt$$
$$\text{s.t.} \quad \dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

The Maximum Principle

$$\begin{aligned} &\max \int_0^T g(x(t), y(t), t)\, dt \\ \text{s.t.} \quad &\dot{x}(t) = h(x(t), y(t), t) \\ &x(0) = x_0 \text{ given} \\ &x(T) \text{ unrestricted.} \end{aligned}$$

Present Value Hamiltonian:

$$H = g(x(t), y(t), t) + \lambda(t) h(x(t), y(t), t)$$

Maximum Principle: Optimal solution $x(t), y(t)$ must satisfy

$$\begin{aligned} &(y) \qquad && H_y = 0 \\ &(x) \qquad && \dot{\lambda}(t) = -H_x \\ &(\lambda) \qquad && \dot{x}(t) = H_\lambda \end{aligned}$$

Exercise: Life-Cycle Consumption

Consider the problem faced by a household that lives from time 0 to time T, and receives instantaneuous labor income every period equal to $w > 0$. Time is continuous. The household can borrow or save at the instantaneous rate $r > 0$. The initial asset position is a_0, and at the end of time (T), the asset position is restricted to be $a(T) \geq 0$. The instantaneous utility function for the household is of the isoelastic form

$$u(c) = \frac{c^{1-\eta} - 1}{1 - \eta}$$

with $\eta > 0$, and the instantaneous subjective discount rate is $\rho > 0$.

1. Write down the objective function of the household. How do we discount future consumption here?
2. Write the budget constraint of the household.
3. With the above you already have the program we want to solve. Write down the Hamiltonian of this program.
4. State the 3 optimal conditions.
5. Find a close form solution.

Detour: From Discrete to Continuous Compounding

Discrete compounding:

$$FV = PV(1 + r)^n$$

- FV: Future Value
- PV: Present Value
- r: Interest rate per compounding period
- n: Number of compounding periods

Example: If you invest \$100 at an annual interest rate of 5% for 3 years, the future value is

$$FV = 100(1 = 0.05)^3 = 115.75$$

Now consider

- r: Annualized interest rate
- n: Number of compounding periods in a year
- $\Delta\tau$: Time horizon

$$FV = PV\left(1 + \frac{r}{n}\right)^{n\Delta\tau}$$

From Discrete to Continuous Compounding

$$FV = PV\left(1 + \frac{r}{n}\right)^{n\Delta\tau}$$

Continuous compounding means $n \to \infty$, so we want to apply $\lim_{n\to\infty}$ to FV.

Note a property of e is:

$$\lim_{m\to\infty}\left(1 + \frac{x}{m}\right)^m = e^x$$

Use $x \equiv r\Delta\tau$, $m \equiv n\Delta\tau$

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$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n\Delta\tau} \equiv \lim_{n\Delta\tau\to\infty}\left(1 + \frac{r\Delta\tau}{n\Delta\tau}\right)^{n\Delta\tau} = e^{r\Delta\tau}$$

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Then

$$\lim_{n\to\infty} FV = PV\ e^{r\Delta\tau}$$

From Discrete to Continuous Compounding

Continuous Compounding:

$$FV = PV \ e^{r\Delta\tau}$$

Applications:

From Discrete to Continuous Compounding

Continuous Compounding:

$$FV = PV\ e^{r\Delta\tau}$$

Applications:

- How much (call it F) will I have in period $t = s$ if I start investing P in period $t = u$, with $s > u$?

$$F = Pe^{\int_u^s r dt} = Pe^{r(s-u)} \quad (= Pe^{r\Delta\tau})$$

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- How much (call it P) can I have in period 0 if in period t I will earn F?

$$P = Fe^{(0-t)r} = Fe^{-rt}$$

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Continuous Compounding:

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$\longrightarrow$ Note: This is the reason why, to evaluate lifetime utility, we discount any period t consumption $c(t)$ with $e^{-\rho t}$.

Life-Cycle Consumption: 1. Household's Objective Function

Hint: Imagine a discrete time problem and make it look continuous.
What do you need to change?

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Hint: Imagine a discrete time problem and make it look continuous.
What do you need to change? (Another hint: $\sum$ and β^t)

Solution:

Life-Cycle Consumption: 2. Budget Constraint

Hint: Start by thinking of the budget of a household that

- Earns wage w instantaneously during that time interval
- Earns interest rate r for her assets which she "brings" from $t + \Delta t$ to t
- Whatever she does not consume, she accumulates as assets for period $t + \Delta t$

Solution:

Life-Cycle Consumption 3: Problem of the HH and Hamiltonian

Hint: For problem

$$\max \int_0^T g(x(t), y(t), t)\, dt$$
$$\text{s.t.} \quad \dot{x}(t) = h(x(t), y(t), t)$$

The Present Value Hamiltonian is:

$$H = g(x(t), y(t), t) + \lambda(t) h(x(t), y(t), t)$$

Here, the Problem of the Household is:

And the Present Value Hamiltonian:

Life-Cycle Consumption 4: State the 3 optimal conditions

Hint: Recall for Present Value Hamiltonian

$$H = g(x(t), y(t), t) + \lambda(t) h(x(t), y(t), t)$$

Maximum Principle: Optimal solution $x(t), y(t)$ must satisfy

$$\begin{array}{ll} (y) & H_y = 0 \\ (x) & \dot{\lambda}(t) = -H_x \\ (\lambda) & \dot{x}(t) = H_\lambda \end{array}$$

Solution:

Life-Cycle Consumption 5: Find a close form solution

Hint:

- First step is solving for $\lambda(t)$ from (a)
- Calculate $\frac{\partial e^{rt}\lambda(t)}{\partial t}$, it could be useful.
- Recall fundamental theorem of calculus

$$\int f'(x)dx = f(x) + const.$$

Life-Cycle Consumption 5: Find a close form solution

Hint:

- Second step is using (c) and what we know of λ to solve for $c(t)$

Life-Cycle Consumption 5: Find a close form solution

Hint:

- Third step is using (λ) and what we know of λ to solve for $c(t)$

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$$
\begin{aligned}
\dot{a}(t) - ra(t) &= w - c(t) \\
e^{-rt}[\dot{a}(t) - ra(t)] &= e^{-rt}[w - c(t)] \\
\frac{\partial e^{-rt}a(t)}{\partial t} &= e^{-rt}[w - c(0)e^{\frac{r-\rho}{\eta}t}] \\
e^{-rt}a(t) &= \int e^{-rt}[w - c(0)e^{\frac{r-\rho}{\eta}t}]dt \\
&= w\int e^{-rt}dt - c(0)\int e^{(\frac{r-\rho}{\eta}-r)t}dt \\
&= w\frac{(-1)}{r}e^{-rt} - c(0)\frac{1}{\frac{r-\rho}{\eta}-r}e^{(\frac{r-\rho}{\eta}-r)t} + B \\
a(t) &= Be^{rt} - \frac{w}{r} - c(0)\frac{1}{\frac{r-\rho}{\eta}-r}e^{(\frac{r-\rho}{\eta})t}
\end{aligned}
$$

Life-Cycle Consumption 5: Find a close form solution

We are almost done, but we still have two unpinned variables B and $c(0)$, so we have to include our final two conditions:

$$[a(0) = a_0] \longrightarrow a_0 = B - \frac{w}{r} - c(0)\frac{1}{\frac{r-\rho}{\eta} - r}$$

$$[a(T) = 0] \longrightarrow 0 = Be^{rT} - \frac{w}{r} - c(0)\frac{1}{\frac{r-\rho}{\eta} - r}e^{(\frac{r-\rho}{\eta})T}$$

This is just a system of 2 equations with 2 unknowns, so we should be able to get the final solutions from here.

Life-Cycle Consumption 5: Find a close form solution

If we multiply the second equation by e^{-rT} and substract both we can get after some algebra that:

$$c(0) = \frac{\frac{r-\rho}{\eta} - r}{e^{(\frac{r-\rho}{\eta}-r)T} - 1}[a_0 + \frac{w}{r}(1 - e^{-rT})]$$

Now that we have $c(0)$, we know all the path of consumption since we expressed $c(t)$ as a function of $c(0)$ and parameters. You could also use $c(0)$ to find B, and use that to derive the optimal asset path. I will leave that as an exercise to you. The final answer should be:

$$a(t) = a_0 e^{rt} + \frac{w}{r}(e^{rt} - 1) - \frac{e^{(\frac{r-\rho}{\eta}-r)t} - 1}{e^{(\frac{r-\rho}{\eta}-r)T} - 1}[a_0 + \frac{w}{r}(1 - e^{-rt})]$$

Note that this formula implies $a(0) = a_0$ and $a(T) = 0$ as it should.
PS: Thanks Facundo for notes!!