Econ 210A
Discussion Section N°8

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Plan for Today

Review of Aiyagari Model

Practice Question: Final 2024 - Neoclassical Model with Employment Risk

Practice Question: Final 2023 - Neoclassical Model with a Borrowing Constraint

Appendix: Invariant distribution of a Markov chain and more

Aiyagari Model

1. **Heterogeneous agents**: Each agent experiences an idiosyncratic income shock s_t
2. **Incomplete markets**: Risk is uninsurable and agents face a borrowing limit $a_{t+1} \geq -b$
3. Need for a new concept of equilibrium: Stationary equilibrium

   defined by $\exists$ of **stationary distribution** $\lambda(a, s)$

$\rightarrow$ Results in:

- Decentralized equilibrium is inefficient because markets are incomplete
- Self-insurance $\implies$ over accumulation of capital
- Aiyagari (1994) meant to characterize extent of precautionary savings in the economy

Plan for Review: Aiyagari Model

We will cover the model by diving into its 3 **building blocks**

1. Heterogeneous agents
2. Incomplete markets
3. Stationary distribution

+ introducing the complete model along the way

Building Block: 1. Heterogeneous Agents

- Each agent experiences an idiosyncratic labor productivity shock

$$s_t = s \in S$$

governed by a Markov process

  - Event s evolves as Markov Chain Details with transition matrix $\mathcal{P}$, with elements $\pi(s'|s)$
  - Probability that s_{t+1} takes specific value s' only depends of history of events/state s^t by last event s_t realization s

$$Pr(s_{t+1} = s'|s^t) = Pr(s_{t+1} = s'|s_t = s) \equiv \pi(s'|s)$$

  - Example: let $s \in S = \{0, 1\}$, then

$$\mathcal{P} = \begin{bmatrix} \pi(0|0) & \pi(1|0) \\ \pi(0|1) & \pi(1|1) \end{bmatrix}$$

Building block: 1. Heterogeneous Agents

Each agent/household chooses $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize expected utility

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right] \equiv \sum_{t=0}^{\infty}\beta^t \sum_{s^t} u(c_t(s^t))\pi(s^t)$$

with budget constraint: $$c_t + a_{t+1} \leq w_t s_t + (1+r_t)a_t \qquad \forall\, s^t, t$$

Recursive formulation:

$$v(a,s;w,r) = \max_{c,a'}\{u(c) + \beta\mathbb{E}_t[v(s',a';w',r')]\}$$

with budget constraint: $$c + a' \leq ws + (1+r)a$$

Note
$$\mathbb{E}_t[v(a',s')] = \mathbb{E}[v(a',s')|a,s] = \sum_{s'} v(a',s')\pi(s'|s)$$

Building Block: 2. Incomplete Markets

Each agent/household solves:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t))\pi(s^t)$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t s_t + (1 + r_t) a_t$$

$$a_{t+1} \geq -b$$

$$a_0, s_0 \quad \text{given}$$

where $s_t \in S$ follows a Markov process: $Pr(s_{t+1} = s' \mid s_t = s) \equiv \pi(s' \mid s)$

Recursive formulation:

$$v(a, s; w, r) = \max_{c, a'} \left\{ u(c) + \beta \sum_{s'} v(a', s'; w', r')\pi(s'|s) \right\}$$

$$\text{s.t.} \quad c + a' \leq ws + (1 + r)a$$

$$a' \geq -b$$

- state: a, s (aggregate state: r, w)
- choice: c, a' policy $a' = g(a, s; w, r)$

Building Block: 3. Stationary Distribution

- Define probability distribution of state (a, s)

$$\lambda_t(a, s) \equiv Pr(a_t = a, s_t = s)$$

- Note policy $a' = g(a, s; r, w)$ induces $\lambda_{t+1}(a', s')$

$$\lambda_{t+1}(a', s') = \sum_{s \in S} \sum_{a : a' = g(a,s)} \lambda_t(a, s)\pi(s'|s)$$

- Stationary distribution: $\lambda(a, s)$ is fix point of

$$\lambda_{t+1} = \mathcal{H}(\lambda_t)$$

$$\implies \lambda^* = \mathcal{H}(\lambda^*)$$

- Interpretation of $\lambda(a, s)$:
  1. Agent level: fraction spent in (a, s) in lifetime
  2. Cross-section: fraction of agents in (a, s) at any time

Aggregation and averages

Because of LLN

- Average wealth in the economy

$$\mathbb{E}[a(r)] = \sum_{s\in S}\sum_{a\in A}\lambda(a,s)g(a,s;r)$$

- Is also the aggregate capital supply in the economy

$$K^s(r) = \sum_{s\in S}\sum_{a\in A}\lambda(a,s)g(a,s;r)$$

* In lecture slides this object is called $A(r)$.

Firm's Problem

$$\max_{K,L} F(K,L) - (r+\delta)K - wL$$

FONC determines optimality

$$\begin{aligned} &\text{Labor demand } L^d: && w = F_L(K,L) \\ &\text{Capital demand } K^d: && r = F_K(K,L) - \delta \end{aligned}$$

Recursive Competitive Equilibrium (RCE)

i. Value fn $v(a, s; \lambda)$, policy fns $a' = g(a, s; \lambda), c = c(a, s; \lambda)$

ii. Firm demands $L^d(\lambda)$, $K^d(\lambda)$

iii. Prices $r(\lambda), w(\lambda)$

iv. A joint distribution $\lambda(a, s)$

v. A perceived law of motion for aggregate state $\lambda'(a', s') = G[\lambda(a, s)]$

such that:

1. Given iii., i. solves the HH problem
2. Given iii., ii. solves the firm's problem
3. Markets clear
4. Expectations are correct: what they perceive is the true law of motion of λ

$$\lambda'(a', z') = G[\lambda(a, z)] = H[\lambda(a, z)]$$

Stationary Recursive Competitive Equilibrium (SRCE)

i. Value fn $v(a,s)$, policy fns $a' = g(a,s), c = c(a,s)$
ii. Firm demands L^d, K^d
iii. Prices r, w
iv. Stationary distribution $\lambda(a,s)$

such that:

1. Given iii., i. solves the HH problem
2. Given iii., ii. solves the firm's problem
3. Markets clear

$$\text{Labor:} \quad L^d = \Pi S$$

Labor supply

$$\text{Capital:} \quad K^d = K^s$$

$$\text{Assets:} \quad K^d = K^s$$

$$\text{Goods:} \quad \sum_s \sum_a \lambda(a,s)c(a,s) + \delta K = F(L,K)$$

4. Stationary distribution $\lambda(a,s)$ solves

$$\lambda(a',s') = \sum_{s \in S} \sum_{a : a' = g(a,s)} \lambda(a,s)\pi(s'|s) \quad \forall a', s'$$

Comments

1. $v(a, z)$
   - In the RCE definition we used λ to characterize the aggregate state instead of w, r

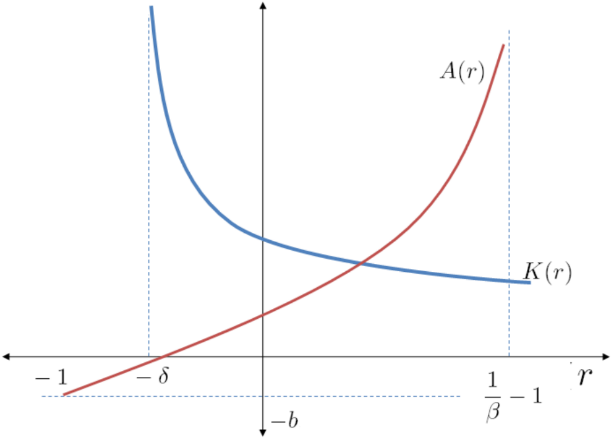
$$v(a, z; \lambda) \equiv v(a, z; w, r)$$

   - We focus on the Stationary Recursive Competitive Equilibrium (SRCE), where λ is constant
   - By consequence, $r(\lambda) = r$ and $w(\lambda) = w$ are fixed
   - In stationary equilibrium, then, it is ok to just keep track of $v(a, z)$
2. SRCE
   - In a SRCE $K = K'$, $A = A'$
   - Solution for the SRCE is finding λ^* and r^* s.t.:

$$\lambda^* = \mathcal{H}(\lambda^*) \tag{1}$$

$$K(r^*) = A(r^*; \lambda^*) \tag{2}$$

Characterizing the SRCE



- $A(r) = \sum_{a,z} \lambda(a,z) g(a,z;r)$

$$\lim_{r \to -\infty} A(r) = -b$$

$$\lim_{r \to \frac{1}{\beta}-1} A(r) = \infty$$

- $K(r) : r = F_K - \delta$

$$\lim_{r \to \infty} K(r) = 0$$

$$\lim_{r \to -\delta} K(r) = \infty$$

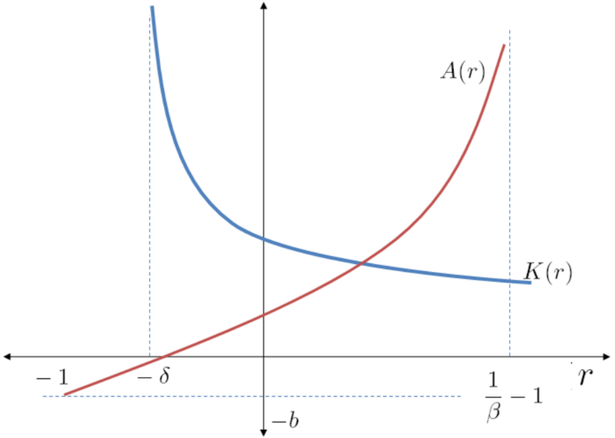
Practice Question

Using the equilibrium of $K(r)$ and $A(r)$

1. Show how equilibrium capital and interest rate would react to a stricter borrowing limit $-b' > -b$
2. Characterize equilibrium capital and interest rate when $-b \to -\infty$
3. Explain how the result of 2. relates to the complete markets allocation

Practice Question

1. Show how equilibrium capital and interest rate would react to a stricter borrowing limit $-b' > -b$



Practice Question

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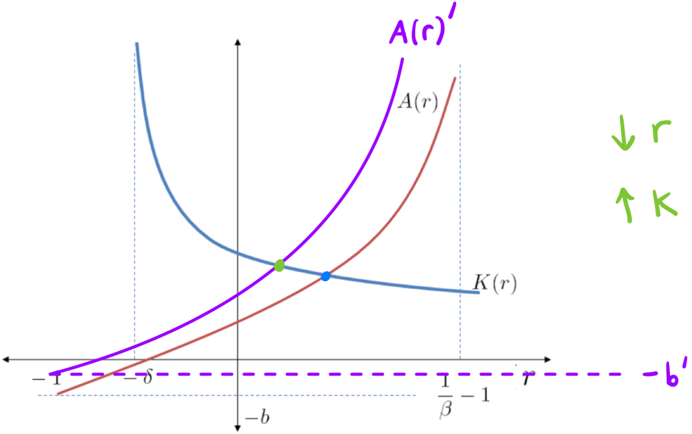
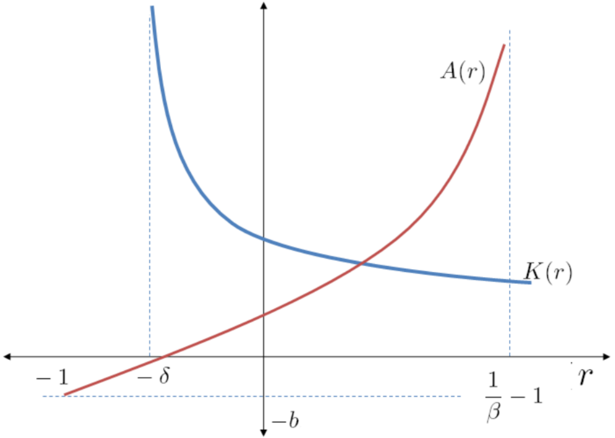


$\downarrow r$

$\uparrow K$

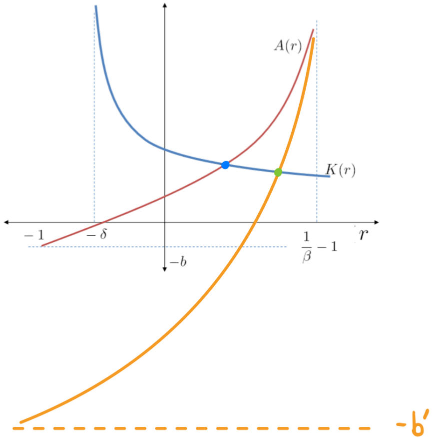
Practice Question

2. Characterize equilibrium capital and interest rate when $-b \to -\infty$



Practice Question

2. Characterize equilibrium capital and interest rate when $-b \to -\infty$



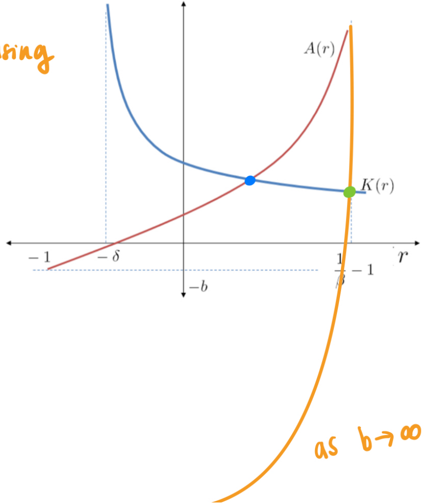
$\uparrow r$

$\downarrow K$

Practice Question

2. Characterize equilibrium capital and interest rate when $-b \to -\infty$

as $b \to \infty$, r is increasing and K is falling



$r^I \to r^c$

$K^I \to K^c$

$$r^c = \frac{1}{\beta} - 1$$

ss rate in neoclassical model

- $F_K = r + \delta$
- $F_K'(K^{ss}) = \frac{1}{\beta} - (1-\delta)$

Final 2024 - Neoclassical Model with Employment Risk

Consider an economy populated by a continuum of ex-ante identical households.
Each household chooses consumption c_t and savings a_{t+1} to maximize expected lifetime utility,

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

subject to a household budget constraint and borrowing limit b,

$$c_t + a_{t+1} \leq w_t s_t + (1 + r_t) a_t$$

$$a_{t+1} \geq -b$$

where s_t represents idiosyncratic employment risk

$$s_t = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p \end{cases}$$

so that $s_t = 1$ represents employment and $s_t = 0$ unemployment.
You may assume that wages w_t and interest rates r_t are fixed.

Final 2024 - Neoclassical Model with Employment Risk

1. Suppose initially that there is no borrowing limit ($b = \infty$) or income uncertainty ($s_t = \bar{s}$). Using the envelope condition, derive the household Euler Equation.
2. Set b to be the "natural borrowing limit". Show how the household Euler Equation changes in this case. Explain.
3. Now consider the full model where –in addition to the borrowing limit– households face stochastic employment risk s_t. Provide an expression for the household Euler Equation in this case.
   Explain how consumption-savings behavior changes relative to the cases in (2) and (1).
4. Is the decentralized equilibrium efficient? Justify your answer.
5. Suppose now that there exists a competitive insurance sector which sells actuarially fair employment insurance. Does this change your answer to part (4)? Explain.

Final 2024 - (1) No income uncertainty, no borrowing limit

Sequential problem under no borrowing limit or income uncertainty

$$\max_{\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$
$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t \bar{s} + (1 + r_t) a_t$$

Assuming wages and interest rates are fixed.

In recursive formulation

$$\begin{aligned} v(a) &= \max_{c, a'} \left\{ u(c) + \beta v(a') \right\} \\ \text{s.t.} & \quad c + a' \leq w \bar{s} + (1 + r) a \end{aligned}$$

Final 2024 - (1) No income uncertainty, no borrowing limit

Lagrangian:

$$\mathcal{L}(a) = u(c) + \beta v(a') + \theta(w\bar{s} + (1+r)a - c - a')$$

FONC:

$$\begin{aligned} \{c\}: \quad & c^{-\gamma} && = \theta \\ \{a'\}: \quad & \beta v'(a') && = \theta \end{aligned}$$

Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \beta v'(a') \tag{1}$$

Envelope:

$$\frac{\partial v(a)}{\partial a} = \theta(1+r) \quad \rightarrow \quad \frac{\partial v(a')}{\partial a'} = \theta'(1+r')$$

Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v'(a') = \frac{\partial v(a')}{\partial a'} = c'^{-\gamma}(1+r')$$

Replace $v'(a')$ in (1) to arrive to Euler Equation:

$$c^{-\gamma} = \beta c'^{-\gamma}(1+r') \tag{2}$$

Final 2024 - (2) No income uncertainty, borrowing limit

Set b to be the "natural borrowing limit". Show how the household Euler Equation changes in this case.

Sequential problem under borrowing limit and no income uncertainty

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \\
\text{s.t.} \quad c_t + a_{t+1} &\leq w_t \bar{s} + (1 + r_t) a_t \\
a_{t+1} &\geq -b
\end{aligned}$$

In recursive formulation

$$\begin{aligned}
v(a) \quad &= \quad \max_{c, a'} \left\{ u(c) + \beta v(a') \right\} \\
\text{s.t.} \quad & \quad c + a' \leq w \bar{s} + (1 + r) \\
& \quad a' \geq -b
\end{aligned}$$

Wages and the interest rate are assumed constant (makes sense given SRCE).

Final 2024 - (2) No income uncertainty, borrowing limit

Lagrangian:

$$\mathcal{L}(a) = u(c) + \beta v(a') + \theta(w\bar{s} + (1+r)a - c - a') + \mu(a' + b)$$

$$\begin{aligned} \text{FONC} \quad \{c\}: & \quad c^{-\gamma} &= \theta \\ \{a'\}: & \quad \beta v'(a') + \mu &= \theta \end{aligned}$$

Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \beta v'(a') + \mu \tag{3}$$

Envelope does not change:

$$\frac{\partial v(a')}{\partial a'} = \theta'(1+r')$$

Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v'(a') = \frac{\partial v(a')}{\partial a'} = c'^{-\gamma}(1+r')$$

Replace $v'(a')$ in (3) to arrive to Euler Equation:

$$c^{-\gamma} = \beta c'^{-\gamma}(1+r') + \mu \tag{4}$$

Final 2024 - (2) No income uncertainty, borrowing limit

Explain how/why EE changes

By complementary slackness condition: $\mu(a' + b) = 0$.

If the constraint does not bind, i.e. $a'(a) > -b$, then $\mu = 0$ and

$$c^{-\gamma} = \beta c'^{-\gamma}(1 + r') \tag{5}$$

Otherwise ($a'(a) \leq b$), $\mu \geq 0$ and

$$c^{-\gamma} \geq \beta c'^{-\gamma}(1 + r') \tag{6}$$

Meaning the agent consumes less today than optimal (higher marginal utility of consumption today than would hold if EE held with equality), only because their optimal level of borrowing is off-limits.

Not because of precautionary savings.

Final 2024 - (3) Income uncertainty, borrowing limit

Sequential problem under borrowing limit and income uncertainty

$$
\begin{aligned}
\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} & \quad \mathbb{E} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \\
\text{s.t.} \quad c_t + a_{t+1} & \leq w_t s_t + (1 + r_t) a_t \\
a_{t+1} & \geq -b
\end{aligned}
$$

In recursive formulation

$$
\begin{aligned}
v(a, s) \quad = & \quad \max_{c, a'} \{u(c) + \beta \mathbb{E} v(a', s')\} \\
\text{s.t.} & \quad c + a' \leq ws + (1 + r) \\
& \quad a' \geq -b
\end{aligned}
$$

Wages and the interest rate are assumed constant (makes sense given SRCE).

Final 2024 - (3) Income uncertainty, borrowing limit

Note

$$\mathbb{E}v(a', s') = pv(a', 1) + (1 - p)v(a', 0)$$

Where $v(a, 1)$:

$$\begin{aligned} v(a,1) &= \max_{c,a'} \{u(c) + \beta \mathbb{E}v(a', s')\} \\ \text{s.t.} \quad & c + a' \leq w + (1 + r) \\ & a' \geq -b \end{aligned}$$

And $v(a, 0)$:

$$\begin{aligned} v(a,0) &= \max_{c,a'} \{u(c) + \beta \mathbb{E}v(a', s')\} \\ \text{s.t.} \quad & c + a' \leq (1 + r) \\ & a' \geq -b \end{aligned}$$

Final 2024 - (3) Income uncertainty, borrowing limit

Lagrangian:

$$\mathcal{L}(a,s) = u(c) + \beta\mathbb{E}v(a',s') + \theta(ws + (1+r)a - c - a') + \mu(a'+b)$$

$$\text{FONC} \quad \begin{aligned} \{c\}: &\quad c^{-\gamma} &= \theta \\ \{a'\}: &\quad \beta\mathbb{E}v_a(a',s') + \mu &= \theta \end{aligned}$$

Combining FONC $\{c\}$ and $\{a'\}$:

$$c^{-\gamma} = \mathbb{E}v_a(a',s') + \mu \tag{7}$$

Envelope:

$$\frac{\partial v(a',s')}{\partial a'} = \theta'(1+r')$$

Replace θ' from FONC $\{c\}$ evaluated at c' to get:

$$v_a(a',s') = \frac{\partial v(a',s')}{\partial a'} = c'^{-\gamma}(1+r')$$

Replace $v_a(a',s')$ in (7) to arrive to Euler Equation:

$$c^{-\gamma} = \beta\mathbb{E}[c'^{-\gamma}(1+r')] + \mu \tag{8}$$

Final 2024 - (3) Income uncertainty, borrowing limit

Explain how/why EE changes

By complementary slackness condition: $\mu(a' + b) = 0$.
If the constraint does not bind, i.e. $a'(a, s) > -b$, then $\mu = 0$ and

$$c^{-\gamma} = \beta\mathbb{E}[c'^{-\gamma}(1 + r')] \tag{9}$$

Note that this does not mean the household choice for consumption and savings is the same as (1) and (2).

Even if $\bar{s} = \mathbb{E}(s) = p$ (so the type in the previous questions is the expected productivity), $u'(\mathbb{E}(s') + (1 + r)a') = u'(\bar{s} + (1 + r)a') < \mathbb{E}(u'(s' + (1 + r)a'))$ because the utility function exhibits Prudence/convex marginal utility (i.e. $u''' > 0$).

Precautionary savings appear to avoid the chances of an event with very low utility in the future, so the household will save more or borrow less even if $\mu = 0$ -constraint doesn't bind-.

Final 2024 - (3) Income uncertainty, borrowing limit

... Explain how/why EE changes

Otherwise ($a'(a,s)^* \leq b$), $\mu \geq 0$ and

$$c^{-\gamma} \geq \beta \mathbb{E} c'^{-\gamma}(1+r') \tag{10}$$

Meaning the constrained agent of course consumes less today than optimal (higher marginal utility of consumption today that would hold if the equation held with equality), because their optimal level of borrowing is off-limits and also because of precautionary savings.

Again, the result is higher savings (less borrowing) than (1) and (2).

Final 2024 - Efficiency under borrowing limit and uncertainty

4. Is the decentralized equilibrium efficient?

   No. There will be higher savings (and therefore higher capital) than in the complete markets equilibrium. The interest rate will also be lower than the efficient one.

5. Does fair insurance change your answer to (4)?

   Yes. The existence of actuarially fair insurance completes the market and therefore this formulation of competitive equilibrium has the efficient capital market allocation.

Final 2023 - Neoclassical Model with a Borrowing Constraint

Consider the problem of a representative household choosing consumption c_t, hours worked n_t, and savings a_{t+1} to maximize expected lifetime utility,

$$\max_{c_t, n_t, a_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \psi \frac{n_t^{1+\nu}}{1+\nu} \right]$$

subject to a household budget and borrowing constraint b,

$$c_t + a_{t+1} \leq w_t z_t n_t + (1 + r_t) a_t$$

$$a_{t+1} \geq -b$$

where $z_t \in Z$ represents a labor productivity shock which is realized at the start of each period and follows a markov process,

$$\mathrm{P}(z_{t+1} = z' | z^t) = \mathrm{P}(z_{t+1} = z' | z_t = z) \equiv \pi(z'|z)$$

The set of potential labor productivity realizations Z is finite, with $\min\{Z\} = 0$, interpreted as non-employment, and $\max\{Z\} < \infty$. Assume that $\pi(z'|z) > 0$ for all $z', z \in Z$.

Final 2023 - Neoclassical Model with a Borrowing Constraint

1. Formulate the Bellman equation for the household problem.
2. Suppose initially that $b = \infty$, so there is no limit on household borrowing. Using the envelope condition, derive two formulas governing optimal household consumption-savings and labor supply decisions.
3. Provide a *brief* interpretation of each formula derived in (2). Explain the economic forces which determine household labor supply and consumption-savings choices.
4. Derive the "natural borrowing limit". Provide intuition for why this is a sensible value to set for constraint b.
   Hint: See lecture 1 of Aiyagari Model.
5. Rederive the optimality conditions in (2) in the presence of the natural borrowing limit. How does household consumption, savings, and labor supply change? Provide intuition.
6. Suppose the household receives a one-time, unanticipated government transfer $\tau > 0$. All else equal, explain how the household's response to the transfer depends on the presence of a borrowing constraint.

Final 2023 - Neoclassical Model with a Borrowing Constraint

1. Formulate the Bellman equation for the household problem.

$$v(a, z; \lambda) = \max_{c,n,a'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} - \psi \frac{n_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t v(a', z'; \lambda') \right\}$$

$$\begin{aligned}
\text{s.t.} \quad & c + a' \leq w(\lambda) z n + (1 + r(\lambda)) a \\
& a' \geq b \\
& \lambda' = G(\lambda) \\
& z \quad \text{Markov chain}
\end{aligned}$$

Where

$$\mathbb{E}_t v(a', z'; \lambda') = \mathbb{E}[v(a', z'; \lambda')|z] = \sum_{z'} \pi(z'|z) v(a', z'|\lambda')$$

Final 2023 - Neoclassical Model with a Borrowing Constraint

2. Suppose initially that $b = \infty$, so there is no limit on household borrowing. Using the envelope condition, derive two formulas governing optimal household consumption-savings and labor supply decisions.

$$\mathcal{L} = \frac{c^{1-\gamma}}{1-\gamma} - \psi \frac{n_t^{1+\nu}}{1+\nu} + \beta \sum_{z'} \pi(z'|z) v(a', z'|\lambda') + \theta(w(\lambda)zn + (1 + r(\lambda))a - c - a')$$

$$\begin{aligned}
\text{FONC} \quad \{c\}: &\quad c^{-\gamma} &= \theta \\
\{n\}: &\quad \psi n^{\nu} - \theta w(\lambda) z &= 0 \\
\{a'\}: &\quad \beta \textstyle\sum_{z'} \pi(z'|z) v_a(a', s') &= \theta
\end{aligned}$$

Envelope

$$\frac{\partial v(a, z; \lambda)}{\partial a} = \theta(1 + r(\lambda)) = c^{-\gamma}(1 + r(\lambda))$$

Final 2023 - Neoclassical Model with a Borrowing Constraint

2. Suppose initially that $b = \infty$, so there is no limit on household borrowing. Using the envelope condition, derive two formulas governing optimal household consumption-savings and labor supply decisions.

$$
\begin{aligned}
EE: \quad & c^{-\gamma} &&= \beta \sum_{z'} \pi(z'|z)(c')^{-\gamma}(1 + r(\lambda')) \\
LL: \quad & \psi n^{\nu} &&= c^{-\gamma} w(\lambda) z
\end{aligned}
$$

Final 2023 - Neoclassical Model with a Borrowing Constraint

3. Provide a *brief* interpretation of each formula derived in (2). Explain the economic forces which determine household labor supply and consumption-savings choices.

    - **Euler-Equation**: At the optimal point, the marginal utility of consumption today must be equal to discounted marginal utility of consuming the earnings of savings of one unit of forgone consumption tomorrow. Otherwise, you maximize consumption by moving consumption toward tomorrow or today through using the financial markets.
    - **Labor-Leisure Condition**: At the optimal point, the marginal disutility of labor must be equal to the (negative of the) marginal utility of consuming the marginal benefit of labor (i.e. the wage).

Final 2023 - Neoclassical Model with a Borrowing Constraint

4. Derive the "natural borrowing limit". Provide intuition for why this is a sensible value to set for constraint b.
   Hint: See lecture 1 of Aiyagari Model.

   The natural borrowing limit is a theoretical justification for the existence of a borrowing limit when there is income uncertainty. The idea is solving for b such that consumers are able to pay back their debt even if they were to experience the worse possible productivity realization $z_t = \underline{z}$ for every period. Because $c_t \geq 0$ we have:

$$c_t = w_t z_t n_t + (1 + r_t)a_t - a_{t+1} \geq 0$$

   Replace $r_t = r$ and $z_t = \underline{z} \qquad \forall t$:

$$a_t \geq -\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t w_t \underline{z} + \left(\frac{1}{1+r}\right)^T a_{T+1}$$

$$\lim_{T\to\infty} a_t \geq \lim_{T\to\infty} -\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t w_t \underline{z} + \left(\frac{1}{1+r}\right)^T a_{T+1} = -\frac{1+r}{r} w_t \underline{z} = 0$$

   Then $a_t \geq 0$, meaning $-b = 0$ is the natural borrowing limit.

Final 2023 - Neoclassical Model with a Borrowing Constraint

5. Re-derive the optimality conditions in (2) in the presence of the natural borrowing limit. How does household consumption, savings, and labor supply change? Provide intuition.

$$\mathcal{L} = \frac{c^{1-\gamma}}{1-\gamma} - \psi\frac{n_t^{1+\nu}}{1+\nu} + \beta\sum_{z'}\pi(z'|z)v(a', z'|\lambda') + \theta(w(\lambda)zn + (1+r(\lambda))a - c - a') + \mu(a' + b)$$

$$\begin{aligned}
\text{FONC} \quad \{c\}: &\quad c^{-\gamma} &= \theta \\
\{n\}: &\quad \psi n^{\nu} - \theta w(\lambda)z &= 0 \\
\{a'\}: &\quad \beta\textstyle\sum_{z'}\pi(z'|z)v_a(a', s') + \mu &= \theta
\end{aligned}$$

Envelope

$$\frac{\partial v(a, z; \lambda)}{\partial a} = \theta(1 + r(\lambda)) = c^{-\gamma}(1 + r(\lambda))$$

Final 2023 - Neoclassical Model with a Borrowing Constraint

5. Rederive the optimality conditions in (2) in the presence of the natural borrowing limit. How does household consumption, savings, and labor supply change? Provide intuition.

$$
\begin{aligned}
EE: \quad & c^{-\gamma} &&= \beta \sum_{z'} \pi(z'|z)(c')^{-\gamma}(1 + r(\lambda')) + \mu \\
LL: \quad & \psi n^{\nu} &&= c^{-\gamma} w(\lambda) z
\end{aligned}
$$

Final 2023 - Neoclassical Model with a Borrowing Constraint

5. Rederive the optimality conditions in (2) in the presence of the natural borrowing limit. How does household consumption, savings, and labor supply change? Provide intuition.
   - EE: If the constraint does not bind, $\mu = 0$, then the euler equation is identical to the problem with $b = -\infty$. If the constraint binds, however, then the household is not consuming optimally. She would like to borrow more and consuming more at the current period, but she can't, so $c^{-\gamma} > \beta \sum_{z'} \pi(z'|z)(c')^{-\gamma}[1 + r(\lambda')]$.
   - Savings will increase because $u'''(\cdot) > 0$, so the HH is prudent and will want to avoid hitting the borrowing constraint. We call this precautionary savings.
   - LL: The labor leisure condition does not change. However, the equilibrium labor supply actually will change since with the borrowing constraint there will be more precautionary savings, so the marginal utility of consumption today will be higher. Therefore the marginal value of work will be higher through the labor-leisure condition, leading workers to supply more labor. Essentially, there are two ways for households to build a buffer stock with the precautionary motive: forgo consumption and work more to boost earnings.

Final 2023 - Neoclassical Model with a Borrowing Constraint

6. Suppose the household receives a one-time, unanticipated government transfer $\tau > 0$. All else equal, explain how the household's response to the transfer depends on the presence of a borrowing constraint.

   Suppose that a household faces a borrowing constraint and that constraint binds. Then that household is consuming less than optimally and any additional transfer will be entirely consumed, which implies that:

   $$\frac{dC_t}{d\tau} = 1$$

   Conversely, suppose does not face a budget constraint or that constraint does not bind. Then that household is consuming optimally and will smooth consuming across periods. Any transfer will be divided across its lifetime, which implies that:

   $$\frac{dC_t}{d\tau} < 1$$

Appendix

Invariant Distribution of Markov Chain

Markov Process

* From Alexis Toda Mathematics for Economists

- When a random variable is indexed by time, we call it a stochastic process.
- For a stochastic process $\{X_t\}_{t=0}^{\infty}$, when the distribution of X_t conditional on past information $X_{t-1}, X_{t-2}, \dots$ depends only on the most recent past (X_{t-1}), we say $\{X_t\}$ is a Markov process.
- When a Markov process $\{X_t\}$ takes on finitely many values, it is called a finite-state Markov chain.
- Let $\{X_t\}$ be a (finite-state) Markov chain and $n = 1, \dots, N$ index values $\{x_n\}_{n=1}^{N}$ the process can take.
- We write $X_t = x_n$ when the state at t is n.
- The Markov chain is completely characterized by the transition probability matrix $\mathcal{P} = (\pi_{nn'})$, where $\pi_{nn'} = \pi(n'|n)$ is the probability of transitioning from state n to n'.
- Note $\pi_{nn'} \geq 0$ and $\sum_{n'=1}^{N} \pi_{nn'} = 1$.

Markov Chain: Invariant Distribution

Theorem: Invariant distribution of a Markov Chain

Let $\mathcal{P} = (\pi_{nn'})$ be a stochastic matrix such that $\pi_{nn'} = \pi(n'|n) > 0$ for all n, n'.
Then there exists a unique invariant distribution Π such that $\Pi = \Pi\mathcal{P}$, and $\lim_{t\to\infty} \mu\mathcal{P}^t = \Pi$ for all initial distributions μ.

From Alexis Toda Mathematics for Economists

Towards a Labor Supply

- Each agent experiences an idiosyncratic labor productivity shock

$$s_t = s \in S$$

  governed by a Markov process
  - State s evolves as Markov Chain with transition matrix $\mathcal{P}$, with elements $\pi(s'|s)$
  - Probability that s_{t+1} takes specific value s' only depends of history/event s^t by last state s_t realization s

$$P(s_{t+1} = s'|s^t) = P(s_{t+1} = s'|s_t = s) \equiv \pi(s'|s)$$

- Time Invariant Distribution: (details in previous slide) $\Pi = \Pi\mathcal{P} \iff (I - \mathcal{P})\Pi = 0 \quad (1)$
  - Example: let $s \in s = \{0, 1\}$, assume

$$\mathcal{P} = \begin{bmatrix} \pi(0|0) & \pi(1|0) \\ \pi(0|1) & \pi(1|1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  Solving for Π in (1) we get: $\Pi = [0.5 \quad 0.5]$

Towards a Labor Supply

- In last slide, solving for Π in (1) we get: $\Pi = [0.5 \quad 0.5]$
- Aggregate labor supply:

$$L^s = \int_0^1 s_i di = \Pi S$$

  - Example: $S = \{0, 1\}$, $\Pi = [0.5 \quad 0.5]$

$$L = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.5$$