Econ 210A
Discussion Section N°7

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Plan for the Rest of the Quarter

Lectures

- RBC
- Ramsey Policies
- Heterogeneous Agent + Financial Frictions (Aiyagari)
- Microfoundations

HW4 (Due Dec 7)

1. Microfoundations for Aggregate Cobb-Douglas Production Function
2. Labor Non-Convexities and Lotteries
3. Optimal Ramsey Policies
4. Computational Exercise: The Aiyagari Model

Discussions

- Today: RBC, Romer
- Nov 27: No discussion (Thanksgiving) Ramsey video?
- Dec 4: Aiyagari
- TBD: Exam review

Dec 12: Final exam

Plan for today

RBC

Practice Question: Endogenous Growth through Expanding Varieties (Final 2023)

Real Business Cycles: Facts and Basic Model

Business cycles as fluctuations around trend

- Empirical challenge: separating trend, cycle, and noise
- Hodrick–Prescott (HP) filter:
  - Low frequency $\Rightarrow$ growth trend
  - Medium frequency $\Rightarrow$ business cycles
  - High frequency $\Rightarrow$ noise

Stylized facts about business cycles

- Relative volatilities: $\sigma_c < \sigma_y \approx \sigma_n < \sigma_i$
- Sectoral comovement and state persistence
- Labor productivity is procyclical

RBC planner problem

$$\max_{\{c_t, n_t, k_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \psi \log(l_t)\right]$$

$$\text{s.t.} \quad y_t = z_t k_t^{\alpha} n_t^{1-\alpha}, \quad c_t + i_t \leq y_t, \quad k_{t+1} = i_t + (1-\delta)k_t, \quad n_t + l_t = 1$$

- z_t follows AR(1) with drift, long-run growth $\approx \bar{\epsilon}$

Real Business Cycles: Performance

- **Equilibrium conditions**
  - Household Euler equation: $\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left(r_{t+1} + 1 - \delta \right) \right]$
  - Labor supply: $\frac{\psi}{c_t} \frac{1}{1-n_t} = w_t$
  - Firm FOCs: $w_t = (1-\alpha) z_t^{1-\alpha} k_t^{\alpha} n_t^{-\alpha}, \quad r_t = \alpha z_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha}$

- **Stationarization and balanced growth**
  - Define $\tilde{c}_t = c_t / z_t$, $\tilde{k}_t = k_t / z_t$
  - Transformed problem satisfies Stokey–Lucas assumptions $\Rightarrow$ unique stationary equilibrium
  - Balanced growth path: constant $\tilde{c}, \tilde{k}, n$

- **Evaluation**
  - **Successes:** comovement, volatility of c, y, k
  - **Failures:** hours too volatile, investment overly volatile, no unemployment margin
  - **Strong implication:** fluctuations are (constrained) efficient $\Rightarrow$ limited scope for stabilization policy

Plan for today

RBC

Practice Question: Endogenous Growth through Expanding Varieties (Final 2023)

Practice Question: Expanding Varieties (Final 2023)

**Adapted for exposition and learning objectives*

The **household problem** is:

$$\max_{c_t, k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1+r_t)k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

The economy has a representative **final good producer** (FGP) who creates output from labor and intermediates inputs $x(i,t)$, for $i \in [0, M(t)]$, with technology

$$Y(t) = n(t)^{1-\alpha} \int_0^{M(t)} x(i,t)^{\alpha} di$$

Intermediate good $x(i,t)$ is produced by monopolistic firm i (IGP) with technology that transforms one unit of final output into one unit of the intermediate good.

New varieties are discovered through investment in R&D. Total investment $Z(t)$, gives

$$\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$$

The PDV of a new discovery i in period t is given by,

$$V(t) = \sum_{j=0}^{\infty} \frac{\pi_{i,t+j}}{(1+r_{t+j})^j}$$

Practice Question: Expanding Varieties (Final 2023)

1. Characterize the household's optimal choices of consumption, capital, and labor.

Extra A. Write the profit maximization problem of the FGP.

Extra B. Write the profit maximization problem of the IGP.

2. Derive the FGP's demand function for labor n and intermediates goods x.

3. Solve the intermediate good producers problem and show that in equilibrium the owner of each variety makes period profits.

$$\pi^* = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$$

Extra C. Define what constitutes a Sequence of Markets Equilibrium in this economy.

4. Derive the BGP growth rate of consumption and GDP-per-capita.

5. Formulate the Benevolent Social Planner's (BSP) problem.

6. Prove that the decentralized equilibrium is inefficient. Provide explanation for why the BSP allocation differs from the decentralized market outcome.

Solution: 1. Optimal household choices

$$\max_{c_t, k_{t+\mathbf{1}}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1 + r_t) k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

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$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1 + r_t) k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

First note: No disutility from labor so $n_t^* = 1$

Solution: 1. Optimal household choices

$$\max_{c_t, k_{t+\mathbf{1}}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1 + r_t) k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

First note: No disutility from labor so $n_t^* = 1$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_t \left(w_t n_t + (1 + r_{t+1}) k_t + \pi_t - c_t - k_{t+1} \right) \right]$$

$$\begin{aligned} \text{FOC} \qquad c_t \quad &: \quad c_t^{-\gamma} = \lambda_t \\ k_{t+1} \quad &: \quad \lambda_t = \beta \lambda_{t+1} (1 + r_{t+1}) \end{aligned}$$

$$\text{EE} \quad c_t^{-\gamma} \quad = \quad \beta [1 + r_{t+1}] c_{t+1}^{-\gamma}$$

Solution: Profit maximization problem of FGP (A) and IGP (B)

FGP The final good producer (FGP) profit maximization problem is given by

$$\Pi(t) = \max_{n(t),\{x(i,t)\}_{i=\mathbf{0}}^{M(t)}} n(t)^{1-\alpha} \int_0^{M(t)} x(i,t)^{\alpha} di - w(t)n(t) - \int_0^{M(t)} p(i,t)x(i,t)$$

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IGP The period t profit maximization problem of the intermediate good producer (IGP) i is given by

$$\pi(i,t) = \max_{x(i,t),p(i,t)} \; p(i,t) \cdot x(i,t) - x(i,t)$$

Note: IGP Technology & Marginal Cost.

- **Baseline**: technology is linear, 1 unit of final good $\Rightarrow$ 1 unit of intermediate (mg cost $= 1$). Total cost of producing $x(i,t)$ units is $x(i,t)$.
- **HW3 Problem 2**: to produce $x(i,t)$ units of intermediate good, i needs $\frac{1}{2}(1 + x(i,t)^2)$ units of capital (mg cost $= \frac{1}{2}(1 + x(i,t)^2) r_t$).

Solution: 2. FGP's demand function for n and $x(i)$

$$\max_{n(t),\{x(i,t)\}_{i=0}^{M(t)}} n(t)^{1-\alpha}\int_0^{M(t)}[x(i,t)]^\alpha di - w(t)n(t) - \int_0^{M(t)} p(i,t)x(i,t)di$$

$$\begin{aligned} \text{FOC} \qquad n(t) \quad &: \quad (1-\alpha)n(t)^{-\alpha}\int_0^{M(t)}[x(i,t)]^\alpha di = w(t) \\ x(i,t) \quad &: \quad \alpha n(t)^{1-\alpha}x(i,t)^{\alpha-1} = p(i,t) \end{aligned}$$

which results in the demand functions:

$$\begin{aligned} n(t) &= (1-\alpha)^{\frac{1}{\alpha}} w(t)^{-\frac{1}{\alpha}} \left[\int_0^{M(t)}[x(i,t)]^\alpha di\right]^{\frac{1}{\alpha}} \\ x(i,t) &= \alpha^{\frac{1}{1-\alpha}} n(t)p(i,t)^{-\frac{1}{1-\alpha}} \end{aligned}$$

Solution: 3. Solve the IGP i problem

$$\pi(i,t) = \max_{x(i,t),p(i,t)} \quad p(i,t) \cdot x(i,t) - x(i,t)$$

Solution: 3. Solve the IGP i problem

$$\pi(i,t) = \max_{x(i,t),p(i,t)} \quad p(i,t)\cdot x(i,t) - x(i,t)$$

IGP i internalizes the FGP demand for their variety

$$\text{FGP FOC} \quad \{x(i,t)\}: \qquad p(i,t) = \alpha n(t)^{1-\alpha} x(i,t)^{\alpha-1} := p(x(i,t))$$

$\star$ **Note:** you can replace $p(x)$ and choose x or $x(p)$ and choose p.

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$$\begin{aligned}
\pi(i,t) &= \max_{x(i,t)} \; p(x(i,t))\cdot x(i,t) - x(i,t) \\
&= \max_{x(i,t)} \; \alpha n(t)^{1-\alpha} x(i,t)^{\alpha-1}\cdot x(i,t) - x(i,t) \\
&= \max_{x(i,t)} \; \alpha n(t)^{1-\alpha} x(i,t)^{\alpha} - x(i,t)
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\end{aligned}$$

$$\text{FOC} \quad x(i,t) \quad : \quad \alpha^2 n(t)^{1-\alpha} x(i,t)^{\alpha-1} - 1 = 0$$

Solution: 3. Solve the IGP i problem

$$\pi(i,t) = \max_{x(i,t),p(i,t)} \quad p(i,t)\cdot x(i,t) - x(i,t)$$

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&= \max_{x(i,t)} \; \alpha n(t)^{1-\alpha} x(i,t)^{\alpha} - x(i,t)
\end{aligned}$$

$$\text{FOC} \quad x(i,t) \quad : \quad \alpha^2 n(t)^{1-\alpha} x(i,t)^{\alpha-1} - 1 = 0$$

In CE $n_t^* = 1$, solving for $x(i,t)$: $\qquad x(i,t) = \alpha^{\frac{2}{1-\alpha}}$

Replace in demand to get: $\qquad p(i,t) = \alpha\left(\alpha^{\frac{2}{1-\alpha}}\right)^{\alpha-1} = \frac{1}{\alpha}$

Solution: 3. Solve the IGP i problem

Same result if IGP internalizes $x(p(i,t))$ and chooses $p(i,t)$,

$$\begin{aligned}\pi(i,t) &= \max_{p(i,t)} \; p(i,t)\cdot x(p(i,t)) - x(p(i,t)) \\ &= \max_{p(i,t)} \; x(p(i,t))[p(i,t)-1]\end{aligned}$$

From FGP FONC

$$x(i,t) = \alpha^{\frac{1}{\alpha-1}} n(t) p(i,t)^{\frac{1}{\alpha-1}} := x(p(i,t))$$

Then

$$\pi(i,t) = \max_{p(i,t)} \; \alpha^{\frac{1}{\alpha-1}} n(t) p(i,t)^{\frac{1}{\alpha-1}} [p(i,t)-1]$$

FONC

$$\begin{aligned} p(i,t) \quad &: \quad \frac{1}{\alpha-1}\alpha^{\frac{1}{\alpha-1}} n(t) p(i,t)^{\frac{1}{\alpha-1}-1}[p(i,t)-1] + \alpha^{\frac{1}{\alpha-1}} n(t) p(i,t)^{\frac{1}{\alpha-1}} = 0 \\ &: \quad \frac{1}{\alpha-1} p(i,t)^{-1}[p(i,t)-1] + 1 = 0 \\ &: \quad 1 - p(i,t)^{-1} + \alpha - 1 = 0 \\ &: \quad p(i,t) = \frac{1}{\alpha}\end{aligned}$$

Solution: 3. Solve the IGP i problem

Profits:

$$
\begin{aligned}
\pi(i,t) = p(i,t)\cdot x(i,t) - x(i,t) &= x(i,t)(p(i,t)-1) \\
&= \alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right) \\
&= \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha)
\end{aligned}
$$

Solution: 3. Solve the IGP i problem

Profits:

$$\begin{aligned}\pi(i,t) = p(i,t)\cdot x(i,t) - x(i,t) &= x(i,t)(p(i,t)-1)\\ &= \alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\\ &= \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha)\end{aligned}$$

Note:

$$\begin{aligned}x(i,t) = \alpha^{\frac{2}{1-\alpha}} &= \bar{x}\\ p(i,t) = \frac{1}{\alpha} &= \bar{p}\\ \pi(i,t) = \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha) &= \bar{\pi}\end{aligned}$$

Solution: C. Define a SoM Equilibrium

- We are asked to compare the Competitive Equilibrium and Benevolent Social Planner allocation.
- You might wonder what is the definition of the CE here.
- The next few slides provide the definition of the Sequence-of-Markets (SoM) Equilibrium.

SoM Equilibrium

A sequence of market equilibrium in this economy consists of paths for prices $\{w_t, r_t, \{p(i,t)\}_{i=0}^{M_t}\}_{t=0}^{\infty}$ and quantities $\{c_t, n_t, k_{t+1}, \{x(i,t)\}_{i=0}^{M_t}, M_{t+1}, Y_t, Z_t\}_{t=0}^{\infty}$ such that:

1. Households maximize utility subject to their budget constraint:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta(1 + r_{t+1})$$

$$n_t = 1$$

2. The final goods producer maximizes profits:

$$n(t) = (1-\alpha)^{1/\alpha} w(t)^{-1/\alpha} \left(\int_0^{M(t)} [x(i,t)]^{\alpha} di\right)^{1/\alpha}$$

$$x(i,t) = \alpha^{1/(1-\alpha)} [p(i,t)]^{-1/(1-\alpha)} n(t)$$

3. Each intermediate goods producer i maximizes profits:

$$x(i,t) = \alpha^{2/(1-\alpha)}$$

$$p(i,t) = 1/\alpha$$

4. Free entry in the R&D sector ensures zero profits: $V_{t+1}(M_{t+1} - M_t) = 1 + r_t$
5. The number of varieties evolves according to: $M_{t+1} - M_t = \eta Z_t$

SoM Equilibrium (Continuation)

Market Clearing Conditions:

1. Labor:

$$\underbrace{n_t}_{n_t^D} = \underbrace{1}_{n_t^S}$$

2. Capital:

$$\underbrace{Z_t}_{k_t^D} = \underbrace{k_t}_{k_t^S}$$

3. Goods:

$$Y_t = C_t + Z_t + \int_0^{M_t} x(i,t)di$$

Solution: 4. Derive the BGP growth rate of consumption and GDP-pc

We will follow 5 steps to get the BGP growth of consumption and GDP-per-capita.

(1) Using EE, find an expression for g_c and a condition on r_{t+1} in the BGP

Solution: 4. Derive the BGP growth rate of consumption and GDP-pc

We will follow 5 steps to get the BGP growth of consumption and GDP-per-capita.

(1) Using EE, find an expression for g_c and a condition on r_{t+1} in the BGP

- From EE:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta(1 + r_{t+1})$$

  Note that if there exists a BGP for consumption

$$g_c = \frac{c_{t+1}}{c_t} = \left[\beta(1 + r_{t+1})\right]^{\frac{1}{\gamma}}$$

  Then r_{t+1} must be constant

Solution: 4. Derive the BGP growth rate of consumption and GDP-pc

(2) From $V(t)$, and free entry of new ideas, find $\bar{r}$

$$V(t) = \sum_{j=0}^{\infty} \frac{\pi_{i,t+j}}{(1+r_{t+j})^j}$$

- Replacing $\bar{\pi}, \bar{r}$:

$$\begin{aligned}
V(t) &= \sum_{j=0}^{\infty} \frac{\bar{\pi}}{(1+\bar{r})^j} \\
&= \bar{\pi} \sum_{j=0}^{\infty} \frac{1}{(1+\bar{r})^j} \\
&= \bar{\pi} \left(\frac{1}{1 - \frac{1}{1+\bar{r}}} \right) \\
&= \bar{\pi} \left(\frac{1+\bar{r}}{\bar{r}} \right) = \bar{V}
\end{aligned}$$

Solution: 4. Derive the BGP growth rate of consumption and GDP-pc

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- Replacing $\bar{\pi}, \bar{r}$:

$$\begin{aligned} V(t) &= \sum_{j=0}^{\infty} \frac{\bar{\pi}}{(1+\bar{r})^j} \\ &= \bar{\pi} \sum_{j=0}^{\infty} \frac{1}{(1+\bar{r})^j} \\ &= \bar{\pi} \left(\frac{1}{1-\frac{1}{1+\bar{r}}} \right) \\ &= \bar{\pi} \left(\frac{1+\bar{r}}{\bar{r}} \right) = \bar{V} \end{aligned}$$

- Free entry implies: $V(t+1)\eta Z_t - (1+r)Z_t = 0 \implies \bar{V}\eta = 1+\bar{r}$

Solution: 4. Derive the BGP growth rate of consumption and GDP-pc

(2) From $V(t)$, and free entry of new ideas, find $\bar{r}$

- Replacing $\bar{V} = \bar{\pi}\left(\frac{1+\bar{r}}{\bar{r}}\right)$

$$\begin{aligned}
\bar{V}\eta &= 1+\bar{r} \\
\bar{\pi}\left(\frac{1+\bar{r}}{\bar{r}}\right)\eta &= 1+\bar{r} \\
\bar{\pi}\left(\frac{1}{\bar{r}}\right)\eta &= 1 \\
\bar{\pi}\eta &= \bar{r}
\end{aligned}$$

- In the BGP, the interest rate will be

$$\begin{aligned}
\bar{r} &= \bar{\pi}\eta \\
&= \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha)\eta
\end{aligned}$$

Solution: 4. Derive the BGP growth rate of C and Y

(3) From $\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$ find a relationship btw g_M and g_Z in BGP

Solution: 4. Derive the BGP growth rate of C and Y

(3) From $\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$ find a relationship btw g_M and g_Z in BGP

- It must be that $g_M = g_Z$

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(4) Derive g_Y and find its relationship with g_M

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Note

$$\begin{aligned} Y(t) &= n(t)^{1-\alpha} \int_0^{M(t)} \bar{x}^{\alpha} di \\ &= n(t)^{1-\alpha} \bar{x}^{\alpha} M(t) \\ &= \bar{x}^{\alpha} M(t) \end{aligned}$$

Solution: 4. Derive the BGP growth rate of C and Y

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\end{aligned}
$$

- $g_Y = \frac{Y(t+1)}{Y(t)} = \frac{\bar{x}^{\alpha} M(t+1)}{\bar{x}^{\alpha} M(t)} = g_M$

Solution: 4. Derive the BGP growth rate of C and Y

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- $g_Y = \frac{Y(t+1)}{Y(t)} = \frac{\bar{x}^{\alpha} M(t+1)}{\bar{x}^{\alpha} M(t)} = g_M$

$\longrightarrow$ In these 4 steps we found an expression for g_c, proved $g_M = g_Z$ and $g_Y = g_M$ (so $g_Y = g_M = g_Z$)

Solution: 4. Derive the BGP growth rate of C and Y

(3) From $\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$ find a relationship btw g_M and g_Z in BGP

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Note

$$\begin{aligned} Y(t) &= n(t)^{1-\alpha} \int_0^{M(t)} \bar{x}^\alpha di \\ &= n(t)^{1-\alpha} \bar{x}^\alpha M(t) \\ &= \bar{x}^\alpha M(t) \end{aligned}$$

- $g_Y = \frac{Y(t+1)}{Y(t)} = \frac{\bar{x}^\alpha M(t+1)}{\bar{x}^\alpha M(t)} = g_M$

$\longrightarrow$ In these 4 steps we found an expression for g_c, proved $g_M = g_Z$ and $g_Y = g_M$ (so $g_Y = g_M = g_Z$)

(5) Finally, use good market clearing to find a relationship btw g_Y and g_c

Solution: 4. Derive the BGP growth rate of C and Y

(3) From $\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$ find a relationship btw g_M and g_Z in BGP

- It must be that $g_M = g_Z$

(4) Derive g_Y and find its relationship with g_M

Note

$$\begin{aligned} Y(t) &= n(t)^{1-\alpha} \int_0^{M(t)} \bar{x}^\alpha di \\ &= n(t)^{1-\alpha} \bar{x}^\alpha M(t) \\ &= \bar{x}^\alpha M(t) \end{aligned}$$

- $g_Y = \frac{Y(t+1)}{Y(t)} = \frac{\bar{x}^\alpha M(t+1)}{\bar{x}^\alpha M(t)} = g_M$

$\longrightarrow$ In these 4 steps we found an expression for g_c, proved $g_M = g_Z$ and $g_Y = g_M$ (so $g_Y = g_M = g_Z$)

(5) Finally, use good market clearing to find a relationship btw g_Y and g_c

Good market clears: $Y_t = C_t + X_t + Z_t$

Solution: 4. Derive the BGP growth rate of C and Y

(5) Use good market clearing to find a relationship btw g_Y and g_c

- Good market clears: $Y_t = C_t + X_t + Z_t$

$$\begin{aligned} Y_t &= C_t + X_t + Z_t && (1) \\ &= C_t + \int_0^{M(t)} x(i,t)di + Z_t && (2) \\ &= C_t + \bar{x}M(t) + Z_t && (3) \end{aligned}$$

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- Reordering

$$
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C_t &= Y_t - \bar{x}M(t) - Z_t\\
C_{\underline{0}}g_c^t &= Y_{\underline{0}}g_Y^t - \bar{x}M(\underline{0})g_M^t - Z_{\underline{0}}g_Z^t
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- From (3) note: $Y_{\underline{0}} = C_{\underline{0}} + \bar{x}M(\underline{0}) + Z_{\underline{0}}$

Solution: 4. Derive the BGP growth rate of C and Y

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- From (3) note: $Y_{\underline{0}} = C_{\underline{0}} + \bar{x}M(\underline{0}) + Z_{\underline{0}}$
  $\rightarrow C_{\underline{0}} = Y_{\underline{0}} - \bar{x}M(\underline{0}) - Z_{\underline{0}}$
- Concluding $g_Y = g_C$

Practice Question: Final 2023 - Expanding Varieties

1. Characterize the household's optimal choices of consumption, capital, and labor.
2. Derive the FGP's demand function for labor n and intermediates goods x.
3. Solve the intermediate good producers problem and show that in equilibrium the owner of each variety makes period profits.
$$\pi^* = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$$
4. Derive the BGP growth rate of consumption and GDP-per-capita.
5. Formulate the Benevolent Social Planner's (BSP) problem.
6. Prove that the decentralized equilibrium is inefficient. Provide explanation for why the BSP allocation differs from the decentralized market outcome.

Solution: 5. Benevolent Social Planner's Problem

BSP maximizes lifetime discounted utility s.t. resource constraint and production technologies

$$\max_{\{c_t, Z_t, n_t, \{x(i,t)\}_{i=0}^{M_t}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$Y_t = n_t^{1-\alpha} \int_0^{M(t)} [x(i,t)]^{\alpha} di \tag{1}$$

$$M_{t+1} = \eta Z_t + M_t \tag{2}$$

$$Y_t = C_t + X_t + Z_t \tag{3}$$

Solution: 5. Benevolent Social Planner's Problem

In HW3, Problem 2 language:

We know efficiency dictates cost minimization in every productive technology

$$\min_{\{x(i,t)\}_{i=0}^{M(t)},M(t)} \int_0^{M(t)} x(i,t)di$$

$$\text{s.t.} \quad Y(t) \leq n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^{\alpha} di$$

$$\mathcal{L} = \int_0^{M(t)} x(i,t)di + \lambda(Y(t) - n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^{\alpha} di)$$

$$\begin{aligned}
\text{FOC} \quad x(i,t) \quad &: \quad 1 - \lambda\alpha x(i,t)^{\alpha-1} = 0 \\
&: \quad 1 = \lambda\alpha x(i,t)^{\alpha-1} \\
&: \quad 1 = \lambda\alpha x(M(t),t)^{\alpha-1} \qquad \text{for } i = M(t) \\
M(t) \quad &: \quad x(M(t),t) - \lambda(n(t)^{1-\alpha} x(M(t),t)^{\alpha}) = 0 \\
&: \quad x(M(t),t) = \lambda(n(t)^{1-\alpha} x(M(t),t)^{\alpha})
\end{aligned}$$

$\star$ To learn how to solve for the FONC w.r.t. to $M(t)$ see Leibniz Rule in following slide.

Useful Tool: Leibniz Rule

Leibniz rule is useful for finding $F'(M(t)) = \frac{\partial F(M(t))}{\partial M(t)}$ when/where:

- $F(M(t)) := n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di$
- $F(M(t)) := \int_0^{M(t)} [x(i,t)] di$

Leibniz Rule

Say you have an integral of the form

$$F(x) = \int_{a(x)}^{b(x)} g(x,i)\, di$$

Then the derivative on x is:

$$F'(x) = g(x, b(x)) \cdot b'(x) - g(x, a(x)) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial g(x,i)}{\partial x}\, di$$

In this type of model:

$$F(M_t) = \int_0^{M_t} g(i)\, di$$

$$F'(M_t) = g(M_t) \cdot 1$$

Solution: 5. Benevolent Social Planner's Problem

In lectures' language:
We know efficiency dictates cost minimization in every productive technology, think of the BSP problem of choosing $\{x(i,t)\}_{i=0}^{M_t}$ keeping M_t fixed.

$$\max_{\{x(i,t)\}_{i=0}^{M_t}} n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di - \int_0^{M(t)} x(i,t) di$$

with FONC:

$$x(i,t) \quad : \quad \alpha n^{1-\alpha} x(i,t)^{\alpha-1} - 1 = 0 \implies x(i,t) = \alpha^{\frac{1}{1-\alpha}} n$$

Now we know

$$X_t = \int_0^{M(t)} [x(i,t)] di = \int_0^{M(t)} [\alpha^{\frac{1}{1-\alpha}} n] di = \alpha^{\frac{1}{1-\alpha}} n M(t)$$

$$Y_t = n^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di = n^{1-\alpha} \int_0^{M(t)} [\alpha^{\frac{1}{1-\alpha}} n]^\alpha di = \alpha^{\frac{\alpha}{1-\alpha}} n M(t)$$

Solution: 5. Benevolent Social Planner's Problem

Now we know

$$X_t = \alpha^{\frac{1}{1-\alpha}} nM(t)$$

$$Y_t = \alpha^{\frac{\alpha}{1-\alpha}} nM(t)$$

- Use this to restate resource constraint (3) $Y_t = C_t + X_t + Z_t$

$$C_t + Z_t = Y_t - X_t = \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t$$

Solution: 5. Benevolent Social Planner's Problem

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- And replace into (2) $M_{t+1} - M_t = \eta Z_t$

Solution: 5. Benevolent Social Planner's Problem

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$$X_t = \alpha^{\frac{1}{1-\alpha}} nM(t)$$

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- And replace into (2) $M_{t+1} - M_t = \eta Z_t$

$$M_{t+1} - M_t = \eta(Y_t - X_t - C_t) = \eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t - \eta C_t$$

Solution: 5. Benevolent Social Planner's Problem

Now we know

$$X_t = \alpha^{\frac{1}{1-\alpha}} nM(t)$$

$$Y_t = \alpha^{\frac{\alpha}{1-\alpha}} nM(t)$$

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$$M_{t+1} = \eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t + M_t - \eta C_t$$

Solution: 5. Benevolent Social Planner's Problem

Now we know

$$X_t = \alpha^{\frac{1}{1-\alpha}} nM(t)$$

$$Y_t = \alpha^{\frac{\alpha}{1-\alpha}} nM(t)$$

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$$M_{t+1} = \eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t + M_t - \eta C_t$$

$$M_{t+1} = \left[\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t$$

Solution: 5. Benevolent Social Planner's Problem

Original BSP:

$$\max_{\{c_t, Z_t, n_t, \{x(i,t)\}_{i=0}^{M_t}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$Y_t = n_t^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di \tag{1}$$

$$M_{t+1} = \eta Z_t + M_t \tag{2}$$

$$Y_t = C_t + X_t + Z_t \tag{3}$$

Now:

$$\max_{\{c_t, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$M_{t+1} = \left[\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t \tag{4}$$

Solution: 5. Benevolent Social Planner's Problem

$$\max_{\{c_t, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$M_{t+1} = \left[\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)n + 1\right] M_t - \eta C_t$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \Lambda_t \left(\left[\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)n + 1\right] M_t - \eta C_t - M_{t+1} \right) \right\}$$

Solution: 5. Benevolent Social Planner's Problem

$$\max_{\{c_t, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$M_{t+1} = \left[\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)n + 1\right] M_t - \eta C_t$$

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FONC:

$$\begin{aligned} C_t &: \quad C_t^{-\gamma} = \Lambda_t \eta \\ M_{t+1} &: \quad \Lambda_t = \Lambda_{t+1}\beta\left[\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)n + 1\right] \end{aligned}$$

Solution: 5. Benevolent Social Planner's Problem

$$\max_{\{c_t, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$M_{t+1} = \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t$$

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$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \Lambda_t \left(\left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t - M_{t+1} \right) \right\}$$

FONC:

$$\begin{aligned} C_t &: \quad C_t^{-\gamma} = \Lambda_t \eta \\ M_{t+1} &: \quad \Lambda_t = \Lambda_{t+1} \beta \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] \end{aligned}$$

Euler Equation:

$$g_c^{sp} \equiv \frac{C_{t+1}}{C_t} = \left(\beta \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] \right)^{\frac{1}{\gamma}}$$

Practice Question: Final 2023 - Expanding Varieties

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2. Derive the FGP's demand function for labor n and intermediates goods x.
3. Solve the intermediate good producers problem and show that in equilibrium the owner of each variety makes period profits.
$$\pi^* = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$$
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5. Formulate the Benevolent Social Planner's (BSP) problem.
6. Prove that the decentralized equilibrium is inefficient. Provide explanation for why the BSP allocation differs from the decentralized market outcome.

Solution: 6. Prove decentralized equilibrium is inefficient

We have arrived to:

1. $g_c^{sp} > g_c$:

$$
\begin{aligned}
g_c^{sp} &= \left(\beta\left(1+\eta\left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\right)\right)^{\frac{1}{\gamma}} \\
&> \left(\beta\left(1+\eta\ \left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\ \alpha^{\frac{1}{1-\alpha}}\right)\right)^{\frac{1}{\gamma}} = g_c
\end{aligned}
$$

2. $\bar{x}^{sp} > \bar{x}$:

$$
\bar{x}^{sp} = \alpha^{\frac{1}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}} = \bar{x}
$$

Solution: 6. Prove decentralized equilibrium is inefficient

We have arrived to:

1. $g_c^{sp} > g_c$:

$$\begin{aligned} g_c^{sp} &= \left(\beta\left(1+\eta\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\right)\right)^{\frac{1}{\gamma}} \\ &> \left(\beta\left(1+\eta\,\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\,\alpha^{\frac{1}{1-\alpha}}\right)\right)^{\frac{1}{\gamma}} = g_c \end{aligned}$$

2. $\bar{x}^{sp} > \bar{x}$:

$$\bar{x}^{sp} = \alpha^{\frac{1}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}} = \bar{x}$$

- In CE, each IGP internalizes the demand they face, which implies they produce smaller amount than efficient, at a higher price ($P > MC$).

Solution: 6. Prove decentralized equilibrium is inefficient

We have arrived to:

1. $g_c^{sp} > g_c$:

$$\begin{aligned} g_c^{sp} &= \left(\beta\left(1+\eta\left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\right)\right)^{\frac{1}{\gamma}} \\ &> \left(\beta\left(1+\eta\ \left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\ \alpha^{\frac{1}{1-\alpha}}\right)\right)^{\frac{1}{\gamma}} = g_c \end{aligned}$$

2. $\bar{x}^{sp} > \bar{x}$:

$$\bar{x}^{sp} = \alpha^{\frac{1}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}} = \bar{x}$$

- In CE, each IGP internalizes the demand they face, which implies they produce smaller amount than efficient, at a higher price ($P > MC$).
- Because profits in the R&D sector depend on $\bar{x}$, less varieties are produced in equilibrium which leads to lower growth.

Solution: 6. Prove decentralized equilibrium is inefficient

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\end{aligned}
$$

2. $\bar{x}^{sp} > \bar{x}$:

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\bar{x}^{sp} = \alpha^{\frac{1}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}} = \bar{x}
$$

- In CE, each IGP internalizes the demand they face, which implies they produce smaller amount than efficient, at a higher price ($P > MC$).
- Because profits in the R&D sector depend on $\bar{x}$, less varieties are produced in equilibrium which leads to lower growth.
- Alternative interpretation: BSP values innovation more because each intermediate good is used more intensively.

Final Comments

- From lecture 11: Policies for restoring efficiency $\rightarrow$ subsidize monopolists (IGP).
- Lecture 16 will introduce *macroeconomic microfoundations*
  $\rightarrow$ Huge gains from caring about what is going on at the decision making level, while keeping tractability for aggregation
- Acemoglu et al. (2018, AER) *microfound* the Romer Model, and get completely different policy recommendations!
  - Incorporate two margins (endogenous exit and reallocation)
  - Subsidizing IGP does not achieve the efficient allocation
  - Taxing operation is the optimal policy
- Main lesson: Market vs efficient allocation comparison is typically a device to show implications of the model. Not public policy recommendations.