ECON 210A: Macroeconomics

Midterm Fall 2025 - Solutions

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1. **[15 pts] Dynamic Programming.** Let $\Big\{\beta, \Gamma, F, X\Big\}$ satisfy assumptions 4.3-4.9 in Stokey & Lucas (S-L). For this question, you may refer to any theorem in S-L without proving it. Consider the sequence problem

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma(x_t) \text{ for all } t$$

$$x_0 > 0 \quad \text{given}$$

Additionally, assume $0 \in \Gamma(x)$ for all $x \in X$. Let $\{x_t^*\}_{t=0}^{\infty}$ be a solution to the problem above.

(A) **[5 pts]** Prove that $F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0$ for all t.

We will prove by contradiction. Without loss of generality[a], assume $\{x_t^*\}_{t=0}^{\infty}$, the solution to the problem, is such that in period t:

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) > 0$$

We take a small variation $\varepsilon > 0$ from x_{t+1}^*, defining $x_{t+1}^\star \equiv x_{t+1}^* - \varepsilon$. By monotonicity of Γ, $x_{t+1}^\star < x_{t+1}^* \implies \Gamma(x_{t+1}^\star) \subset \Gamma(x_{t+1}^*)$ so $x_{t+2}^\star \leq x_{t+1}^*$. Because ε is small, it is safe to assume $x_{t+2}^\star = x_{t+2}^*$.
We will now explore how lifetime utility

$$V = \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*)$$

changes. Since we only changed x_{t+1}^*, V is only affected through $F(x_t, x_{t+1})$ and $F(x_{t+1}, x_{t+2})$. Thus, we can define the change in lifetime utility by

$$D = F(x_t^*, x_{t+1}^*) + \beta F(x_{t+1}^*, x_{t+2}^*) - \big(F(x_t^*, x_{t+1}^\star) + \beta F(x_{t+1}^\star, x_{t+2}^*)\big)$$

Using a first order taylor expansion, we can characterize the new reward function at $t, t+1$ up to first order.

$$F(x_t^*, x_{t+1}^\star) + \beta F(x_{t+1}^\star, x_{t+2}^*) \approx F(x_t^*, x_{t+1}^*) + \beta F(x_{t+1}^*, x_{t+2}^*) + \left.\frac{\partial \left(F(x_t, x_{t+1}) + \beta F(x_{t+1}, x_{t+2})\right)}{\partial x_{t+1}}\right|_{x_t^*, x_{t+1}^*, x_{t+2}^*} (x_{t+1}^\star - x_{t+1}^*)$$

Where

$$\left.\frac{\partial \left(F(x_t, x_{t+1}) + \beta F(x_{t+1}, x_{t+2})\right)}{\partial x_{t+1}}\right|_{x_t^*, x_{t+1}^*, x_{t+2}^*} = F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) \tag{1}$$

Replacing (1) and noticing $x_{t+1}^\star - x_{t+1}^* = -\varepsilon$

$$\begin{aligned} &F(x_t^*, x_{t+1}^\star) + \beta F(x_{t+1}^\star, x_{t+2}^*) \approx \\ &F(x_t^*, x_{t+1}^*) + \beta F(x_{t+1}^*, x_{t+2}^*) - \left(F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*)\right) \varepsilon \end{aligned}$$

Replacing in D:

$$D \approx \left(F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*)\right) \varepsilon$$

By assumption $F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) > 0$ and $\varepsilon > 0$ so $D > 0$. A contradiction with $\{x_t^*\}_{t=0}^\infty$ being a solution. Then it must be the case that the Euler Equation holds $\forall t$.

□

[a] The opposite case will be symmetric.

(B) [**5 pts**] Formulate the problem recursively. What is the problem's state variable?

$$v(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta v(x')\}$$

State variable: x

(C) [**5 pts**] Formally explain what constitutes a solution to the recursive problem. How does it relate to the optimal sequence $\{x_t^*\}_{t=0}^\infty$ solving the problem above? Be precise.

A solution to the recursive problem is given by a value function $v(x)$ and a policy function $g(x) = \{x' \in \Gamma(x) : v(x) = F(x, x' + \beta v(x')\}$.
Under assumptions 4.3-4.9 in S-L, Theorems 4.2 (the value function v of the sequence problem satisfies the FE), 4.3 (partial converse imposing boundedness of v), 4.4 (if $\{x_t^*\}_{t=0}^\infty$ solve the SP, then it also satisfies the FE) and 4.5 (any sequence $\{x_t^*\}_{t=0}^\infty$ that satisfies the FE and boundedness, is a solution to the SP) hold. The four theorems together imply the solutions to the problem defined above –the SP– and the recursive one –FE– coincide exactly and the optimal policies are defined as above.

2. **[85 pts] Poverty Traps and Growth Takeoffs in the Neoclassical Model.** Consider an economy populated by infinitely lived households with forward-looking preferences,

$$V(k_0) \;=\; \max_{c_t,\, k_{t+1},\, n_t} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

The economy's aggregate resource and time budget constraints are given by

$$\begin{aligned} c_t + k_{t+1} &\leq A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t \\ 0 \;\leq\; n_t &\;\leq\; 1 \end{aligned}$$

where $\{\alpha, \delta\} \in (0,1)$, A represents aggregate TFP, and initial capital $k_0 > 0$ is given.

(A) **[5 pts]** Define what constitutes a solution to the benevolent social planner's problem.

We could either define the sequential or recursive solution:

- Sequential: Given k_0, a sequence $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ such that discounted social utility is maximized and the aggregate resource constraints of the economy are satisfied.
- Recursive: Given k_0, a value function $v(k)$ and policy functions $c(k), k'(k)$ that maximize discounted utility subject to $c + k' \leq Ak^{\alpha} + (1-\delta)k$.

$$\begin{aligned} v(k) &= \max_{c,\, k'} \{\log c + \beta\, v(k')\} \\ \text{s.t.} \quad & c + k' \leq Ak^{\alpha} + (1-\delta)k. \end{aligned}$$

Using the fact that $n = 1$ because there is no disutility of labor.

(B) **[5 pts]** Prove that there exists a unique solution to the social planner's problem.

If we haven't already, we need to start by writing down the recursive formulation of the problem. The Bellman equation is:

$$\begin{aligned} v(k) &= \max_{c,\, k'} \{\log c + \beta\, v(k')\} \\ \text{s.t.} \quad & c + k' \leq Ak^{\alpha} + (1-\delta)k. \end{aligned}$$

The solution to the social planner's problem is unique if there exists a unique $v(k)$ that solves the Bellman equation.
Define the Bellman operator $T : C_b \to C_b$ by

$$T[v](k) \;=\; \max_{k'} \left\{ \log\!\big(Ak^{\alpha} + (1-\delta)k - k'\big) + \beta\, v(k') \right\}.$$

We show T is a contraction (modulus β), hence v is the unique fixed point of T, and uniquely solves the Bellman equation.

i. **Monotonicity.** If $v(k') \geq w(k')$ for all k', then for every k',

$$\log\big(Ak^\alpha + (1-\delta)k - k'\big) + \beta v(k') \;\geq\; \log\big(Ak^\alpha + (1-\delta)k - k'\big) + \beta w(k'),$$

so taking maxima over k' yields $T[v](k) \geq T[w](k)$.

ii. **Discounting.** For scalar a,

$$\begin{aligned} T[v+a](k) &= \max_{k'} \big\{\log\big(Ak^\alpha + (1-\delta)k - k'\big) + \beta\big(v(k') + a\big)\big\} \\ &= \max_{k'} \big\{\log\big(Ak^\alpha + (1-\delta)k - k'\big) + \beta v(k')\big\} + \beta a \\ &= T[v](k) + \beta a. \end{aligned}$$

By Blackwell's sufficient conditions, T is a contraction with modulus β, so v is the unique fixed point of T and solves the Bellman equation.

(C) **[10 pts]** Derive the Euler Equation. What does it say about the optimal allocations chosen by the social planner $\{c_t^*, k_{t+1}^*, n_t^*\}$?

We can find the Euler Equation by taking FOC on the sequential or recursive problem.

Recursive:

$$\mathcal{L} = \max_{c,k'} \{\log c + \beta v(k') + \lambda[Ak^\alpha + (1-\delta)k - c - k']\Big\}$$

FOC:

$$\begin{aligned} c &: \;\; \frac{1}{c} - \lambda = 0 \\ k' &: \;\; \beta v'(k') - \lambda = 0 \end{aligned}$$

Envelope:

$$\begin{aligned} v'(k) &= \lambda[\alpha A k^{\alpha-1} + 1 - \delta] \\ v'(k') &= \lambda'[\alpha A (k')^{\alpha-1} + 1 - \delta] \end{aligned}$$

Replace λ from consumption FOC in Envelope and in capital FOC, then Envelope in capital FOC and get:

$$\text{Euler Equation:} \quad \frac{1}{c} = \beta \frac{1}{c'}\Big[\alpha A (k')^{\alpha-1} + 1 - \delta\Big]$$

Interpretation: Optimality in the consumption-savings decision implies the marginal benefit in terms of utility of one extra unit of consumption is set equal to the discounted marginal benefit in terms of utility out of destining the extra unit of consumption, instead, to savings. This is because each extra unit of savings implies $\alpha A(k')^{\alpha-1}$ (MPK) extra units of production tomorrow + the not-depreciated extra capital $1-\delta$.

(D) **[5 pts]** Solve for the steady state level of capital k_{ss}.

In steady state

$$\text{Euler Equation:} \quad \frac{1}{c_{ss}} = \beta \frac{1}{c_{ss}} \Big[\alpha A (k_{ss})^{\alpha-1} + 1 - \delta \Big]$$

$$\begin{aligned}
1 &= \beta \Big[\alpha A (k_{ss})^{\alpha-1} + 1 - \delta \Big] \\
\frac{1}{\beta} &= \alpha A (k_{ss})^{\alpha-1} + 1 - \delta \\
\frac{1}{\beta} - (1-\delta) &= \alpha A (k_{ss})^{\alpha-1} \\
k_{ss} &= \left(\frac{1}{\alpha A} \Big(\frac{1}{\beta} - (1-\delta) \Big) \right)^{\frac{1}{\alpha-1}}
\end{aligned}$$

Equivalent to the k_{ss} result from class:

$$k_{ss} = f'^{-1} \Big(\frac{1}{\beta} - (1-\delta) \Big)$$

(E) **[10 pts]** Prove that the optimal allocations $\{c_t^*, k_{t+1}^*, n_t^*\}$ have the property that they eventually converge to the unique steady state c_{ss}, k_{ss}, n_{ss}.

We can prove that the optimal allocations $\{c_t^*, k_{t+1}^*, n_t^*\}$ converge to the unique steady state c_{ss}, k_{ss}, n_{ss} noting concavity of v implies:

$$[v'(g(k)) - v'(k)]\,[g(k) - k] \leq 0$$

where $g(k) = k'$ is the optimal policy for capital tomorrow given capital today is k. Substituting $v'(g(k)) = v'(k') = \frac{\lambda}{\beta}$ from the FOC and $v'(k) = \lambda(f'(k) + (1-\delta))$ from the Envelope Condition

$$\big[f'(k) + (1-\delta) - \beta^{-1}) \big]\,(g(k) - k) \geq 0$$

Claim: the optimal policy $g(k_t) = k_{t+1}$ is such that k converges monotonically to k_{ss}. This means:

$$\begin{aligned}
k_t > k_{ss} \quad &\Longleftrightarrow \quad k_{ss} < g(k_t) = k_{t+1} < k_t \\
k_t < k_{ss} \quad &\Longleftrightarrow \quad k_t < g(k_t) = k_{t+1} < k_{ss}
\end{aligned}$$

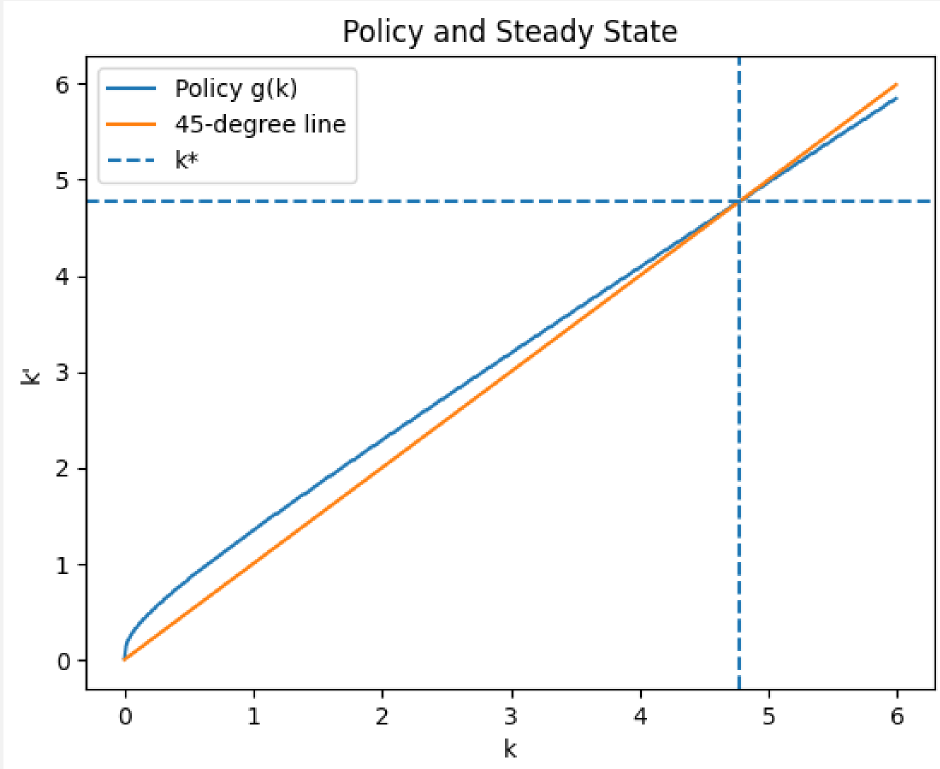
Proof: We know

$$\underbrace{\big[f'(k) - \beta^{-1} + (1-\delta) \big]}_{A} \underbrace{[g(k) - k]}_{B} \geq 0$$

Recall at k_{ss}, $f'(k_{ss}) = \beta^{-1} - (1-\delta)$
By strict concavity of f:

- If $k > k_{ss}$: $f'(k) < \beta^{-1} - (1-\delta)$ $(A < 0)$ so it must be that $g(k) - k < 0$ $(B < 0)$ "capital stock shrinks"
- If $k < k_{ss}$: $f'(k) > \beta^{-1} - (1-\delta)$ $(A > 0)$ so it must be that $g(k) - k > 0$ $(B > 0)$ "capital stock grows"

□



(F) **[5 pts]** Does the economy's initial condition k_0 influence its long-run prospects? Can foreign aid to poor countries deliver sustained increases in welfare? Explain your answer.

No. The economy will converge to the same k_{ss} regardless of k_0.

Now suppose, for the following question parts (G) - (I), that aggregate TFP grows exogenously at rate $1+\gamma$ so that,

$$A_{t+1} = (1+\gamma)A_t$$

(G) **[5 pts]** Define what is meant by balanced growth path. Prove there exists a unique balanced growth path. You may refer to your answers above.

A balance growth path (BGP) is a path for capital, consumption and output, such that all variables are growing at a constant rate.

To prove existence and uniqueness of the BGP, we will map this model to one we already know.

Defining z_t as $A_t = z_t^{1-\alpha}$: $F(k_t) = z_t^{1-\alpha} k_t^{\alpha}$

$$z_{t+1}^{1-\alpha} = (1+\gamma) z_t^{1-\alpha}$$

$$\left(\frac{z_{t+1}}{z_t}\right)^{1-\alpha} = (1+\gamma)$$

$$\frac{z_{t+1}}{z_t} = (1+\gamma)^{\frac{1}{1-\alpha}}$$

Then our problem is equivalent to Solow I with

$$\frac{z_{t+1}}{z_t} = (1+\gamma)^{\frac{1}{1-\alpha}}$$

We could stop here by pointing we know Solow I has a unique BGP with

$$\frac{k_{t+1}}{k_t} = \frac{z_{t+1}}{z_t} = (1+\gamma)^{\frac{1}{1-\alpha}}$$

Or we could continue the proof. Note that if we redefine all variables $\tilde{x}_t = \frac{x_t}{z_t}$ we get a transformed model that satisfies S&L assumptions. This implies the transformed model has a unique steady state defined by

$$\tilde{k}_{ss} = f'^{-1}(\tilde{\beta}^{-1} - (1-\delta))$$

Now note that finding a steady state for $\tilde{k} = k/z$ does not mean our model has a steady state, but the transformed model has a steady state. We know that z is growing over time at rate $(1+\gamma)^{\frac{1}{1-\alpha}}$. Then it must be the case that k is also growing over time, at the exact same rate. More formally:

$$\tilde{k}_{ss} \implies \frac{k_{t+1}}{z_{t+1}} = \frac{k_t}{z_t} \implies \frac{k_{t+1}}{k_t} = \frac{z_{t+1}}{z_t} = (1+\gamma)^{\frac{1}{1-\alpha}}$$

A proof of existence and uniqueness of a BGP.

(H) **[10 pts]** Derive the economy's long-run growth rate of per-capita-GDP $y_t = A_t k_t^{\alpha}$.

We want to find $\frac{y_{t+1}}{y_t}$.

$$\begin{aligned} t: \quad & y_t = A_t k_t^{\alpha} \\ t+1: \quad & y_{t+1} = A_{t+1} k_{t+1}^{\alpha} \\ \text{Dividing:} \quad & \frac{y_{t+1}}{y_t} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}}{k_t}\right)^{\alpha} \end{aligned}$$

Replacing $\frac{k_{t+1}}{k_t} = (1+\gamma)^{\frac{1}{1-\alpha}}$ and $\frac{A_{t+1}}{A_t} = (1+\gamma)$:

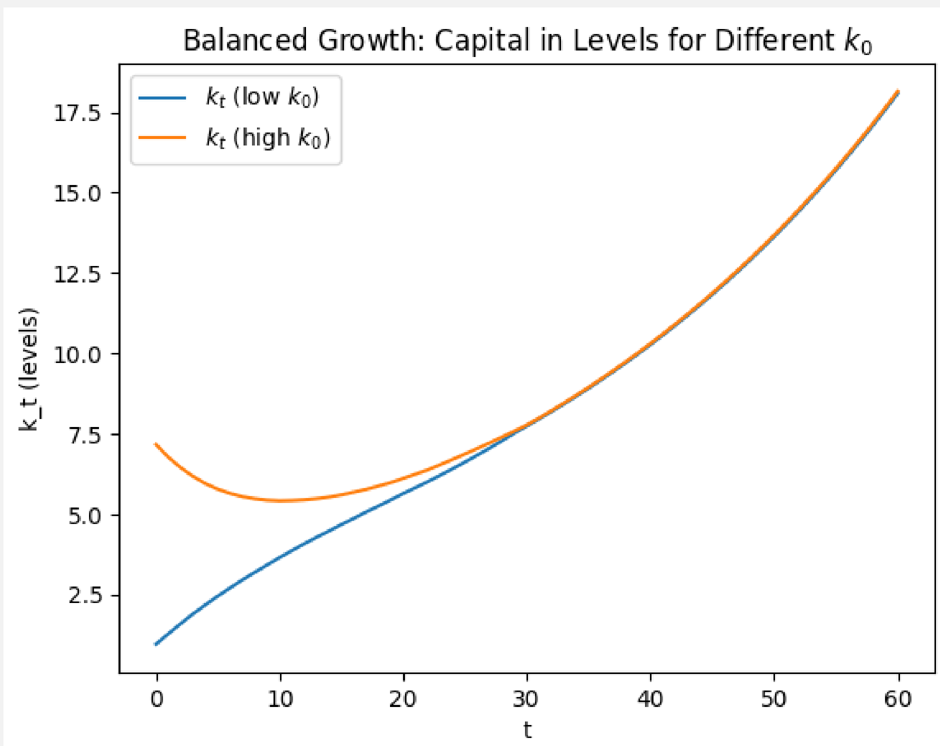
$$\begin{aligned}\frac{y_{t+1}}{y_t} &= (1+\gamma)\left((1+\gamma)^{\frac{1}{1-\alpha}}\right)^{\alpha} \\ &= (1+\gamma)^{\frac{1}{1-\alpha}}\end{aligned}$$

You could also note that this is the Solow I model, where

$$\frac{y_{t+1}}{y_t} = \frac{k_{t+1}}{k_t} = \frac{z_{t+1}}{z_t} = (1+\gamma)^{\frac{1}{1-\alpha}}$$

(I) **[5 pts]** Does the economy's initial condition k_0 influence its long-run prospects? Can foreign aid to poor countries deliver sustained increases in welfare? Explain your answer.

No. Long-run growth is given exogenously by the rate at which A_t grows. Foreign aid will not be able to achieve **sustained increases** in welfare. Some argued that difference in levels could be persistent. However, if an economy start off with a low k_0 it will accumulate capital faster than one with a larger k_0, relative to some optimal k_0 given A_0.



Now suppose, for the remainder of the question, that TFP A_t only grows if firms pay a new technology adoption cost. The cost is given by zA_t, where $z > 0$ is a constant. The cost scaling with current TFP A_t reflects the fact that adopting more advanced technologies is more costly. The firm's production technology in period t is given by,

$$F(k_t \mid A_t) = \begin{cases} (1+\gamma)A_t k_t^\alpha n_t^{1-\alpha} & \text{if innovation cost } zA_t \text{ paid} \\ A_t k_t^\alpha n_t^{1-\alpha} & \text{if no innovation spending} \end{cases}$$

where $\gamma > 0$. If firm's adopt the new technology, they will enter next period with higher TFP. If firms do not pay the innovation cost, their productivity will stagnate so that

$$A_{t+1} = \begin{cases} (1+\gamma)A_t & \text{if R\&D cost } zA_t \text{ paid} \\ A_t & \text{if no R\&D spending} \end{cases}$$

(J) **[5 pts]** Show that the firm's optimal technology adoption decision follows a cut-off rule, whereby the firm optimally pays the innovation cost and adopts the new technology if,

$$k_t \geq \left[\frac{z}{\gamma}\right]^{\frac{1}{\alpha}}$$

When choosing to invest or not, the firms face the following problem

$$\max[(1+\gamma)A_t k_t^\alpha - zA_t; A_t k_t^\alpha]$$

So they will chose to invest as long as

$$\begin{aligned} (1+\gamma)A_t k_t^\alpha - zA_t &\geq A_t k_t^\alpha \\ 1+\gamma - \frac{z}{k_t^\alpha} &\geq 1 \\ \gamma - \frac{z}{k_t^\alpha} &\geq 0 \\ \gamma &\geq \frac{z}{k_t^\alpha} \\ k_t^\alpha &\geq \frac{z}{\gamma} \\ k_t &\geq \left(\frac{z}{\gamma}\right)^{\frac{1}{\alpha}} \equiv \hat{k} \end{aligned}$$

(K) **[10 pts]** Argue that there can exist multiple equilibria in this economy (aside from the trivial equilibrium at $k = 0$), one which is a "poverty trap" in which the economy stagnates and the other a "modern growth" equilibrium with sustained long-run growth. [**HINT:** you may draw a picture and/or refer to answers above to make your argument]

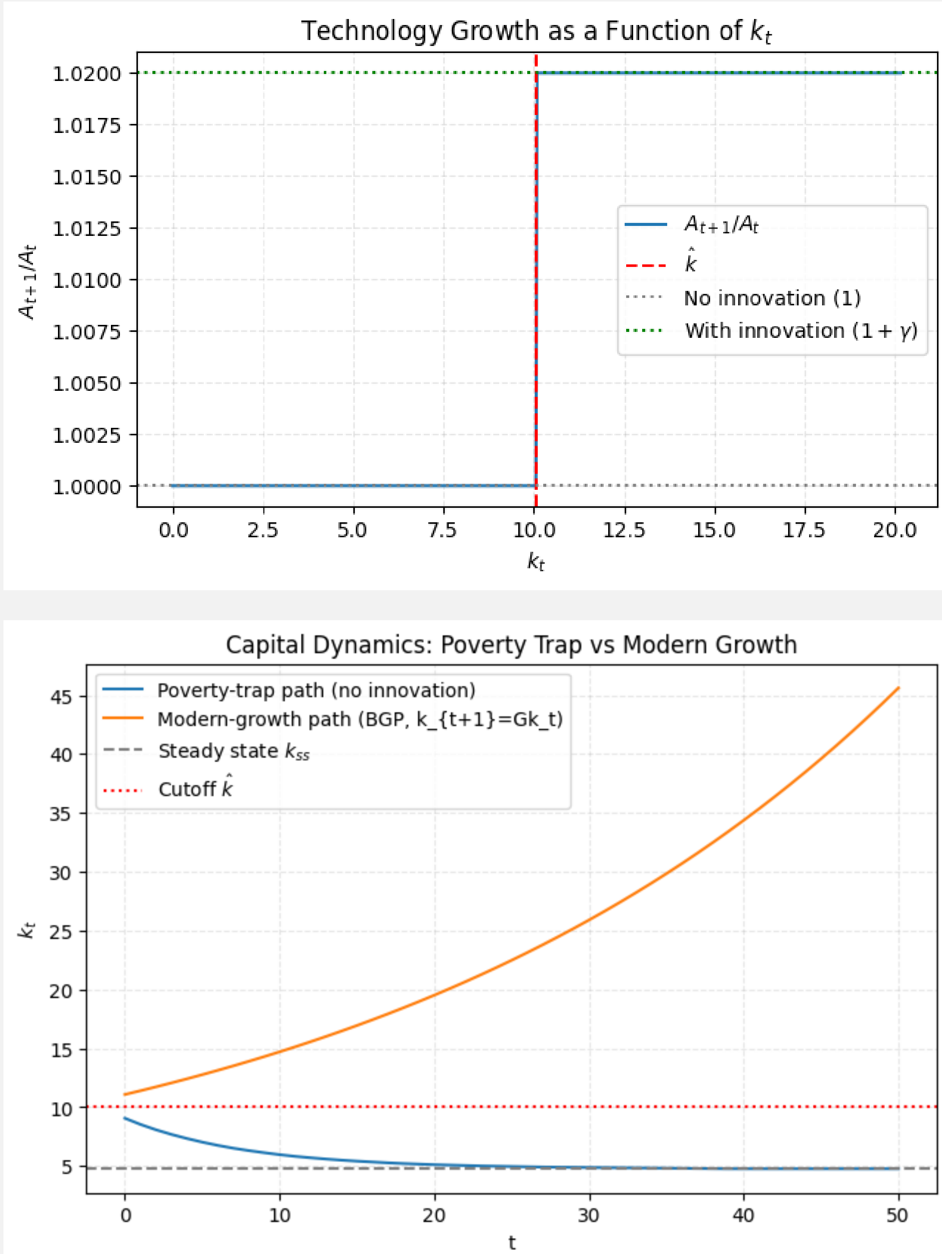
In question (F), the economy will converge to a steady state capital k_{ss} regardless of where it started. In question (I), the economy will grow at rate $(1+\gamma)^{\frac{1}{1-\alpha}}$ regardless of where it started. Here, however, under some conditions (in particular $\hat{k} < k_{ss}$) the initial capital k_0 **will** determine the equilibrium.

To fix ideas, we will explore different possibilities:

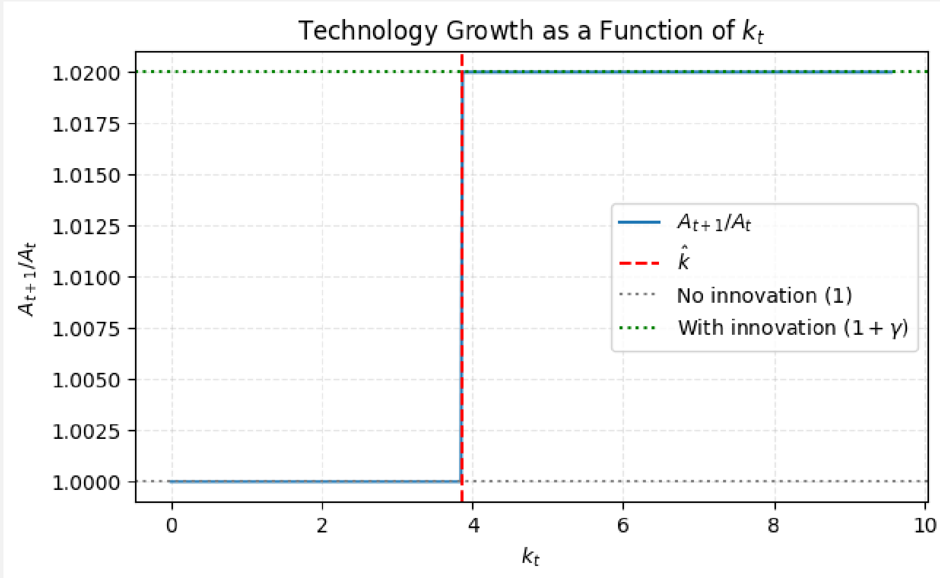
- If $k_0 < k_{ss} < \hat{k}$, the economy will accumulate capital and grow only until it reaches k_{ss}. Within this region for k_0, k_0 does not matter, all economies would converge to the same k_{ss}.
- If $k_{ss} < k_0 < \hat{k}$, the economy will dis-accumulate capital and shrink until it reaches k_{ss}. Within this region for k_0, k_0 does not matter, all economies would converge to the same $k_s s$.
- If $k_{ss} < \hat{k} < k_0$, because k_0 is higher than the threshold for investing in new technologies, the economy will invest. The next period capital, k_1, will also be above the threshold, so investment will materialize again. This means $k_0 > \hat{k}$ allows for the initial investment, but also unlocks long-run growth potential. This economy will be in a BGP.

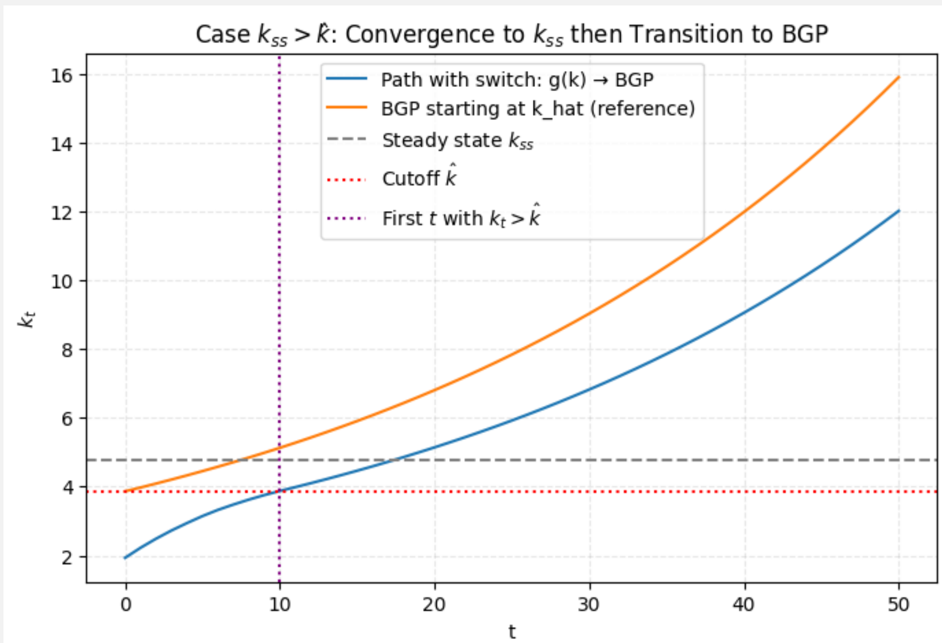
Note that if $\hat{k} < k_{ss}$ the equilibrium is unique. Regardless of where the economy starts k_0, it will converge to k_{ss}. If it is converging from below, then it will reach a point in which capital will reach the threshold and therefore investment will begin. If it is converging from above, the threshold is already satisfied so investments will materialize. The unique equilibrium under $\hat{k} < k_{ss}$ is the BGP.
On the other hand, if $\hat{k} > k_{ss}$, the equilibrium does depend on k_0. If $k_0 > \hat{k}$ the equilibrium will be the BGP. If $k_0 < \hat{k}$, the equilibrium will be the steady state.

Multiple Equilibria: Case where $k_{ss} < \hat{k}$



Unique Equilibrium: Case where $k_{ss} > \hat{k}$





(L) **[5 pts]** Explain what causes the multiplicity of equilibria relative to our neoclassical assumptions that deliver a single unique equilibrium. Which assumption is violated?

Some assumptions that are violated are:

- **A3 (Continuity and compactness of $\Gamma(k_t)$):** violated because $f(k_t)$ is discontinuous $\Rightarrow \Gamma(k_t)$ is not continuous.
- **A8 (Convexity of $\Gamma(k_t)$):** violated because the fixed R&D cost creates a non-convex feasible set. In the neoclassical growth model, the production function is concave $\implies$ convex set.

(M) **[5 pts]** Does the economy's initial condition k_0 influence its long-run prospects? Can foreign aid to poor countries deliver sustained increases in welfare? Explain your answer.

Yes. If $\hat{k} > k_{ss}$, k_0 determines whether the economy will converge to k_{ss} or to a BGP. In this context, foreign aid can help achieve sustained increases in welfare. Think for example in an economy starting at $k_0 = \hat{k} - \varepsilon$. Because $k_{ss} < k_0 = \hat{k} - \varepsilon < \hat{k}$, the economy will start dis-accumulating capital until it reaches k_{ss}. A very small on time aid could allow the economy to start investing and therefore achieving sustained economic growth. The counterfactual is the economy will shrink reaching the steady state, where it will stagnate.

3. **[5 pts] Extra Credit.** We discussed in lecture how extreme value (EV) distributed random variables are a widely used tool in economic modeling. Briefly explain what mathematical property this class of random variables possesses that make them particularly useful for modeling many economic phenomenon.

The difference between two independent EV-distributed random variables follows a logistic distribution, which implies that the probability of choosing one option over another has a closed-form expression given by the logit formula. This allows economists to represent stochastic choice behavior while keeping the model analytically tractable.
EV random variables are either "min stable" or "max stable" and so many economics problems that have discrete optimization (eg, accept or reject a job; purchase or don't purchase a car) are very amenable to including EV idiosyncratic components since they turn these non-convex optimization problems into convex ones by making the solution a continuous probability.