Econ 210A
Discussion Section N°5

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Plan for today

General Midterm Review

Update to Overview of Growth Models from DS4

Practice Question: Dynamic Programming (Midterm 2021)

Practice Question: Endogenous Labor Supply in Neoclassical Growth Model (Midterm 2021)

Practice Question: Endogenous Growth through Expanding Varieties (Final 2023)

What You Need For the Midterm

You will most likely find 3 or 4 questions asking about:

1. **Dynamic programming**
   - Show a unique solution exists (Blackwell) - **DS2**
2. **Some kind of Neoclassical Model**
   - Formulate BSP recursively, define a solution (prove a unique solution exists, Blackwell) - **DS2**
   - Formulate the household problem recursively - **DS3**
   - Using FOC and Envelope, characterize EE, LL - **DS3**
   - Define a SoM equilibrium or RCE - **DS3**
   - Compare the BSP and markets allocations - **DS3**
   - Maybe: Does the model exhibit long-run growth?
3. **Growth**
   - Romer (2021) - Practice question in appendix
   - Two Sector Growth Model (2022)
   - CES Production Technology (2023) - **DS4**
   - Ak Growth with Human Capital (2024) - **DS4**

Tools for Solving a Model

The optimization problem you will need to solve might be:

- A Benevolent Social Planner (BSP) problem
  - Sequential
  - Recursive
- An agent's problem (i.e. Household) in a decentralized equilibrium
  - Sequential
  - Recursive

Yo want to be comfortable with these tools:

- Transform a Sequential Problem (SP) to a Recursive Formulation/Functional Equation/Bellman Equation (FE)
- Find the EE, LL. How?
  (1) Manage your constraints. 2 alternatives:
    1. Replace constraints in the objective function
    2. Set a Lagrangian

  (2) Using FONC, find EE and LL
    - SP: FONC $\Longrightarrow$ EE, LL
    - FE: FONC + Envelope $\Longrightarrow$ EE, LL

Overview Growth Models (From DS4)

Exogenous Growth

Solow I
Neutral TFP growth

$$f(k_t) = z_t^{1-\alpha} k_t^{\alpha}$$

$$z_{t+1} = z_t \exp(\mu)$$

rr_k As $k \to \infty$: $rr_k > 0$

g_k $g_k = g_z = \exp(\mu)$

Solow II
Embodied TFP growth

$$f(k_t) = k_t^{\alpha}$$

$$\begin{aligned} k_{t+1} &= q_{t+1} i_t + (1-\delta) k_t \\ q_{t+1} &= q_t \exp(\mu) \end{aligned}$$

rr_k As $k \to \infty$: $rr_k > 0$

g_k $g_k = \exp\left(\frac{\mu}{1-\alpha}\right)$

Endogenous Growth

AK
No decreasing returns to capital

$$f(k_t) = A k_t$$

rr_k As $k \to \infty$: $rr_k = A - \delta > 0$

g_k $g_k = \left[\beta(A + 1 - \delta)\right]^{1/\gamma}$

More Growth Models

Endogenous Growth

Human Capital

Human capital type models get through decreasing mg returns to physical capital by introducing a second productivity enhancing technology, education

$$Y_t = f(n_t, k_t) = An_t^{1-\alpha} k_t^{\alpha}$$
$$k_{t+1} = i_t + (1-\delta)k_t$$
$$h_{t+1} = h_t + \theta(h_t - n_t)$$

- h_t: effective hours for production and education
- n_t: effective hours destined for production
- $h_t - n_t$: effective hours destined for investing in education

$$rr_k = \frac{MPK + (1-\delta)p'_k}{p'_k} = \alpha \left[\frac{\theta x'}{1+\theta - z'}\right]^{\alpha-1} + (1-\delta)$$
$$rr_h = \frac{MPH + (1-\delta)p'_h}{p'_h} = (1-\alpha)\left[\frac{\theta \cdot x}{1+\theta - z}\right]$$

with $x_t = \frac{k_t}{h_t}$ and $z_t = \frac{h_{t+1}}{h_t}$

More Growth Models

Endogenous Growth

Expanding Varieties

$$Y_t = n_t^{1-\alpha} \int_0^{M_t} x_t(i)^{\alpha} di$$

- rr_k Romer type models get through marginal decreasing returns to investing in capital by expanding varieties M_t.
- g_Y We will actually arrive to a $g_Y = [\beta(\cdot)]^{\frac{1}{\gamma}}$ of the Ak type

Practice Question: Dynamic Programming (Midterm 2021, Final 2024)

For each statement, identify whether the claim is true, false, or uncertain. Justify your answer in a few sentences.

a. Every decision problem can be written recursively.

   False. Problem needs to be of sequential structure and stationary. Counter examples: time inconsistency, history dependence.

b. If the flow payoff $F(x, x')$ is bounded, then there exists a unique bounded solution to the Bellman equation.

   True. A unique solution v exists because the Bellman operator $T : B \to B$ has a unique fixed point $Tv = v$.
   Boundedness of F ensures $T : B \to B$ is a contraction (by Blackwell's conditions), so by the Contraction Mapping Theorem, a unique fixed point — and thus a unique v — exists.

   Extra: Why did we need **A3** in class but not here?
   A3 was used to guarantee $T : C_B \to C_B$ (continuity of the value function) via the Theorem of the Maximum.
   Here we only need $T : B \to B$, which holds trivially when F is bounded.

   A3: X convex, $\Gamma : X \to X$ non-empty, compact, continuous.

Practice Question: Dynamic Programming (Midterm 2021, Final 2024)

c. If the flow payoff function $F(x, x')$ is unbounded, then there does not exist a unique bounded solution to the Bellman Equation.

   False. Unbounded F means the usual sufficient conditions (Blackwell's contraction) do not hold, so we *cannot prove* existence and uniqueness with the same tools.

   However, this does *not* mean a recursive solution cannot exist. The recursive formulation may still be valid and even have a unique solution—just not guaranteed by the Contraction Mapping Theorem.

   If we redefine the state space so that $F : \hat{X} \times \hat{X} \to \mathbb{R}$ becomes bounded (e.g., restrict to a compact subset), the result is recovered.

   Example: The neoclassical growth model. Its payoff function is unbounded, but uniqueness still holds once we restrict feasible capital to the *maximum maintainable capital stock.*

d. Suppose assumptions A3-A9 in Stokey & Lucas hold, then a feasible sequence $\{x_t^*\}_{t=0}^{\infty}$ can be optimal even if

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) \neq 0$$

   False. Stokey & Lucas Theorem 4.15: EE and TC are necessary and sufficient for a solution $\implies$ there cannot be a solution such that EE does not hold.

e. In growth models with neoclassical preferences and technologies, the economy's initial conditions do not matter for its *long-run* growth path.

   False. True for exogenous growth model. Not for Ak or HK.

Practice Question: Endogenous Labor Supply in NGM (Midterm 2021)

Time is discrete, $t = 0, 1, 2, \ldots$. Households $i \in [0,1]$ have preferences:

$$u_i(c, \ell) = \frac{c^{1-\gamma} - 1}{1-\gamma} - \psi \frac{\ell^{1+\phi} - 1}{1+\phi},$$

with consumption c and labor ℓ.

They supply ℓ_t at wage w_t, save assets k_t earning r_t, and face depreciation $\delta \in [0,1]$. Firms are identical and competitive, with technology:

$$Y = F(K, L) = AK^{\theta} L^{1-\theta}.$$

a. Define the Sequence of Markets Equilibrium given k_0.
b. Write the household problem recursively and define the Recursive Competitive Equilibrium.
c. Using either characterization, solve for the market equilibrium. Show that hh allocations can be characterized by two conditions: Euler Equation and the Labor–Leisure trade-off. Explain each.
d. Characterize the steady state. Does the economy exhibit sustained growth? Why or why not?
e. Suppose $A_{t+1} = A_t e^{\mu}$ with $\mu > 0$. Does the model admit a Balanced Growth Path? How does it depend on γ?

Solution a. SoM Equilibrium given k_0

Households' Problem: $\max_{\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma} - 1}{1-\gamma} - \psi \frac{\ell_t^{1+\phi} - 1}{1+\phi} \right]$

$$s.t. \qquad c_t + k_{t+1} \leq r_t k_t + w_t \ell_t + (1-\delta) k_t + \pi_t$$

$$k_0 \text{ given}$$

Firms' Problem: $(\forall t) \quad \pi_t = \max_{\{K_t^d, L_t^d\}} A(K_t^d)^\theta (L_t^d)^{1-\theta} - r_t K_t^d - w_t N_t^d$

A **Sequence-of-Markets equilibrium** consists of:

(i) household choice streams for consumption, work, and savings: $\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}$.

(ii) firms' choice streams for capital and labor demand: $\{K_t^d, L_t^d\}_{t=0}^{\infty}$.

(iii) prices streams: $\{w_t, r_t\}_{t=0}^{\infty}$.

Such that:

(a) Given (iii), (i) solves the households' problem.

(b) Given (iii), (ii) solves the firms' problem.

(c) Markets clear:

- $L_{d,t} = L_t = \int_0^1 \ell_t(i) di = \ell_t$
- $K_{d,t} = K_t = \int_0^1 k_t(i) di = k_t$
- $C_t + K_{t+1} = A(K_t^d)^\theta (L_t^d)^{1-\theta} + (1-\delta) K_t$
- Due to free entry, there are zero profits: $\pi_t = 0$.

Solution b. Write Household Problem Recursively and Define RCE

Transform the problem into a recursive form:

$$
\begin{aligned}
v(K_0, k_0) &\equiv \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}-1}{1-\gamma} - \psi \frac{\ell_t^{1+\phi}-1}{1+\phi} \right] \\
&= \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \left\{ \left[\frac{c_0^{1-\gamma}-1}{1-\gamma} - \psi \frac{\ell_0^{1+\phi}-1}{1+\phi} \right] + \beta \sum_{j=0}^{\infty} \beta^j \left[\frac{c_j^{1-\gamma}-1}{1-\gamma} - \psi \frac{\ell_j^{1+\phi}-1}{1+\phi} \right] \right\} \\
&= \max_{\{c_0,\ell_0,k_1\}} \left\{ \frac{c_0^{1-\gamma}-1}{1-\gamma} - \psi \frac{\ell_0^{1+\phi}-1}{1+\phi} + \beta v(K_1, k_1) \right\}
\end{aligned}
$$

Noting that the starting period is arbitrary this transformation is valid not only for $t = 0, 1$, but for any $t, t+1 \in \{0, 1, \cdots\}$:

$$
v(K, k) = \max_{\{c,\ell,k'\}} \left\{ \frac{c^{1-\gamma}-1}{1-\gamma} - \psi \frac{\ell^{1+\phi}-1}{1+\phi} + \beta v(K', k') \right\}
$$

Solution b. Write Household Problem Recursively and Define RCE

Households' Problem: $$v(K,k) = \max_{\{c,\ell,k'\}} \left\{ \frac{c^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell^{1+\phi}-1}{1+\phi} + \beta v(K',k') \right\}$$

$$\text{s.t.} \quad c + k' \leq w(K)\ell + r(K)k + (1-\delta)k + \pi(K), \qquad K' = \hat{G}(K)$$

Firms' Problem: $$\max_{\{K^d,L^d\}} \pi(K) = A(K^d)^\theta (L^d)^{1-\theta} - r(K)K^d - w(K)N^d$$

A **Recursive Competitive Equilibrium** consists of:

i) a value function $v(K,k)$ and policy functions $c(K,k), k'(K,k), \ell(K,k)$

ii) a perceived law of motion for aggregate capital $\hat{G}(K)$

iii) price functions $w(K), r(K)$

iv) firms policy functions $K_d(K), N_d(K), \pi(K)$

such that:

a) Given (ii) and (iii), (i) solves the household problem

b) Given (iii), (iv) solves the firm's problem

c) Markets clear:
   - $K^d(K) = k(K,K) = \int_0^1 k(K,k)(i)di$
   - $L^d(K) = \ell(K,K) = \int_0^1 \ell(K,k)(i)di$
   - $c(K,K) + k'(K,K) = A[K^d(K)]^\theta [L^d(K)]^{1-\theta} + (1-\delta)K$

d) Expectations are correct: $\hat{G}(K) = k'(K,K)$.

e) Due to free entry, profits are zero $\pi(K) = 0$.

Solution c. Using either characterization, solve for the CE

Our plan is characterize the CE $\implies$ Provide the set of equations that must hold in CE.

Today: RCE. If you want I can upload SoM.

Household Problem:

$$\mathcal{L} = \max_{\{c,\ell,k'\}} \left\{ \frac{c^{1-\gamma}-1}{1-\gamma} - \psi \frac{\ell^{1+\phi}-1}{1+\phi} + \beta v(K',k') + \lambda[r(K)k + w(K)\ell + (1-\delta)k + \pi(K) - c - k'] \right\}$$

FOC:

$$\begin{aligned} c &: \quad c^{-\gamma} - \lambda = 0 \\ \ell &: \quad -\psi \ell^{\phi} + \lambda w(K) = 0 \\ k' &: \quad -\lambda + \beta v_{k'}(K',k') = 0 \end{aligned}$$

Envelope:

$$\begin{aligned} v_k(K,k) &= \lambda[r(K) + 1 - \delta] \\ v_{k'}(K',k') &= \lambda'[r(K') + 1 - \delta] \end{aligned}$$

Solution c. Characterize the RCE

Households' Problem:

Combining the FOCs and the envelope condition, we can derive an Euler Equation (EE) and a labor-leisure (LL) conditions:

$$
\begin{aligned}
\text{EE} &: \quad c^{-\gamma} = \beta[r(K') + (1-\delta)](c')^{-\gamma} \\
\text{LL} &: \quad \psi \ell^{\phi} = c^{-\gamma} w(K)
\end{aligned}
$$

Firms' Problem:

$$
\max_{\{K^d, L^d\}} \pi(K) = A(K^d)^{\theta}(L^d)^{1-\theta} - r(K)K^d - w(K)N^d
$$

$$
\begin{aligned}
K^d &: \quad r(K) = \theta \frac{Y}{K^d} \\
L^d &: \quad w(K) = (1-\theta)\frac{Y}{L^d}
\end{aligned}
$$

Solution d. Characterize the Steady State

We drop subscripts and primes and state that every variable is expressed at the steady-state.
From the Euler Equation, we know that:

$$1 = \beta[r + (1 - \delta)] \implies r = \frac{1}{\beta} - (1 - \delta)$$

From there, we can solve for the capital to output ratio in terms of parameters, using K^d:

$$\theta \frac{Y}{K} = r \implies \frac{K}{Y} = \frac{\theta}{r}$$

such that K/Y is in terms of parameters.

$F(\cdot, \cdot)$ is CRS, by Euler's Theorem, $F_K(K, L)K + F_L(K, L)L = Y \implies rK + wL = Y$.

Rewrite the budget constraint as:

$$C_t + K_{t+1} = r_t K_t + w_t L_t + (1 - \delta)K_t = Y_t + (1 - \delta)K_t$$

In steady state:

$$C + K = Y + (1 - \delta)K \implies \frac{C}{Y} = 1 - \delta \frac{K}{Y}$$

Solution d. Characterize the Steady State

From the labor-leisure condition and the optimality condition for the firm, we know:

$$(1-\theta)\frac{Y}{L} = w, \qquad \psi L^{\phi} = C^{-\gamma} w \implies \psi L^{\phi} = C^{-\gamma}(1-\theta)\frac{Y}{L}$$

Solve for L:

$$L \quad = \quad \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{1+\phi}} Y^{\frac{1-\gamma}{1+\phi}}$$

Replace into the production function:

$$\begin{aligned}
Y &= AK^{\theta}L^{1-\theta} \\
&= \ldots \\
&= \left[A\left(\frac{K}{Y}\right)^{\theta}\right]^{\frac{1+\phi}{(1-\theta)(\phi+\gamma)}} \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{\phi+\gamma}}
\end{aligned}$$

Solution d. Characterize the Steady State

We can now solve for L:

$$L = \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{\phi+\gamma}} \left[A\left(\frac{K}{Y}\right)^{\theta}\right]^{\frac{1-\gamma}{(1-\theta)(\phi+\gamma)}}$$

and w:

$$w = (1-\theta)\frac{Y}{L}$$

Solution d. Characterize the Steady State

We have fully characterized the steady state in terms of parameters:

$$
\begin{aligned}
r &= \frac{1}{\beta} - (1-\delta) \\
\frac{K}{Y} &= \frac{\theta}{r} \\
\frac{C}{Y} &= 1 - \delta \frac{K}{Y} \\
Y &= \left[A \left(\frac{K}{Y} \right)^{\theta} \right]^{\frac{1+\phi}{(1-\theta)(\phi+\gamma)}} \left[\frac{1}{\psi} (C/Y)^{-\gamma} (1-\theta) \right]^{\frac{1}{\phi+\gamma}} \\
L &= \left[\frac{1}{\psi} (C/Y)^{-\gamma} (1-\theta) \right]^{\frac{1}{\phi+\gamma}} \left[A \left(\frac{K}{Y} \right)^{\theta} \right]^{\frac{1-\gamma}{(1-\theta)(\phi+\gamma)}} \\
w &= (1-\theta) \frac{Y}{L}
\end{aligned}
$$

Solution e. Balanced Growth Path

Suppose $A_{t+1} = A_t e^{\mu}$ with $\mu > 0$. Does the model admit a Balanced Growth Path? How does it depend on γ?

- In this case, the problem no longer satisfies the $S\&L$ assumptions since $\Gamma(k)$ is now unbounded.
- Using a change of variables, denote $A_t = Z_t^{1-\theta}$. You will realize that, in that case, the problem is identical to the Solow I model presented in class.
- For full points, it is enough to state the fact above and state the growth rate $\exp(\mu/(1-\theta))$ and to state that the BGP only exists if $\gamma = 1$.

Solution e. Balanced Growth Path: Why $\gamma = 1$?

1st note: $g_L = 0$

Suppose not.

- If $g_L > 0$:

$$L_t = e^{g_L t} L_0 \quad \Rightarrow \quad \lim_{t \to \infty} L_t = \infty,$$

  which is impossible since $L_t \in (0, 1)$.

- If $g_L < 0$:

$$L_t = e^{g_L t} L_0 \quad \Rightarrow \quad \lim_{t \to \infty} L_t = 0,$$

  implying zero production and zero consumption.
  But this cannot happen in equilibrium because:

$$\lim_{c \to 0} u'(c) = \infty \quad \text{(Inada condition).}$$

Solution e. Balanced Growth Path: Why $\gamma = 1$?

First-order condition:

$$\psi L_t^{\phi} = w_t c_t^{-\gamma}$$

Taking logs:

$$\log \psi + \phi \log L_t = \log w_t - \gamma \log c_t$$

and one period earlier:

$$\log \psi + \phi \log L_{t-1} = \log w_{t-1} - \gamma \log c_{t-1}$$

Subtracting:

$$\phi g_L = g_w - \gamma g_c$$

We want $g_L = 0 \quad [\log L_t = \log L_{t-1}]$
Then:

$$g_w = \gamma g_c$$

If we know $g_w = g_c$, it follows that:

$$\gamma = 1$$

Intuition:
Substitution and wealth effects cancel out, so that even if the BGP exhibits increasing wages, labor supply is constant.

Plan for today

General Midterm Review

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Practice Question: Endogenous Growth through Expanding Varieties (Final 2023)

Practice Question: Expanding Varieties (Final 2023)

**Adapted for exposition and learning objectives*

The **household problem** is:

$$\max_{c_t, k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1+r_t)k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

The economy has a representative **final good producer** (FGP) who creates output from labor and intermediates inputs $x(i,t)$, for $i \in [0, M(t)]$, with technology

$$Y(t) = n(t)^{1-\alpha} \int_0^{M(t)} x(i,t)^{\alpha} di$$

Intermediate good $x(i,t)$ is produced by monopolistic firm i (IGP) with technology that transforms one unit of final output into one unit of the intermediate good.

New varieties are discovered through investment in R&D. Total investment $Z(t)$, gives

$$\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$$

The PDV of a new discovery i in period t is given by,

$$V(t) = \sum_{j=0}^{\infty} \frac{\pi_{i,t+j}}{(1+r_{t+j})^j}$$

Practice Question: Expanding Varieties (Final 2023)

1. Characterize the household's optimal choices of consumption, capital, and labor.

(Added) Write the profit maximization problem of the FGP.

(Added) Write the profit maximization problem of the IGP.

2. Derive the FGP's demand function for labor n and intermediates goods x.

3. Solve the intermediate good producers problem and show that in equilibrium the owner of each variety makes period profits.

$$\pi^* = (1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$$

4. Derive the BGP growth rate of consumption and GDP-per-capita.

5. Formulate the Benevolent Social Planner's (BSP) problem.

6. Prove that the decentralized equilibrium is inefficient. Provide explanation for why the BSP allocation differs from the decentralized market outcome.

Solution: 1. Optimal household choices

$$\max_{c_t, k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1 + r_t)k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

First note: No disutility from labor so $n_t^* = 1$
Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_t \left(w_t n_t + (1 + r_{t+1})k_t + \pi_t - c_t - k_{t+1} \right) \right]$$

FONC:

$$\begin{aligned} c_t &: \quad c_t^{-\gamma} = \lambda_t \\ k_{t+1} &: \quad \lambda_t = \beta \lambda_{t+1}(1 + r_{t+1}) \end{aligned}$$

EE:

$$\left(\frac{c_{t+1}}{c_t} \right)^{\gamma} = \beta[1 + r_{t+1}]$$

Solution: Profit maximization problem of FGP and IGP

The final good producer (FGP) profit maximization problem is given by

$$\Pi(t) = \max_{n(t), \{x(i,t)\}_{i=\mathbf{0}}^{M(t)}} n(t)^{1-\alpha} \int_0^{M(t)} x(i,t)^{\alpha} di - w(t)n(t) - \int_0^{M(t)} p(i,t)x(i,t)$$

The period t profit maximization problem of the intermediate good producer (IGP) i is given by

$$\pi(i,t) = \max_{x(i,t), p(i,t)} p(i,t) \cdot x(i,t) - x(i,t)$$

Note: technology transforms 1 unit of final good into 1 unit of intermediate, so i needs $x(i,t)$ units of final good to produce $x(i,t)$ units of intermediate good.
In HW3 Problem 2, to produce $x(i,t)$ units of final good, i needs $\frac{1}{2}(1 + x(i,t)^2)$ units of capital, so mg cost is $\frac{1}{2}(1 + x(i,t)^2)r_t$

Solution: 2. FGP's demand function for n and $x(i)$

$$\max_{n(t),\{x(i,t)\}_{i=0}^{M(t)}} n(t)^{1-\alpha}\int_0^{M(t)}[x(i,t)]^\alpha di - w(t)n(t) - \int_0^{M(t)} p(i,t)x(i,t)di$$

FONC:

$$\begin{aligned} n(t) &: \quad (1-\alpha)n(t)^{-\alpha}\int_0^{M(t)}[x(i,t)]^\alpha di = w(t) \\ x(i,t) &: \quad \alpha n(t)^{1-\alpha}x(i,t)^{\alpha-1} = p(i,t) \end{aligned}$$

which results in the demand functions:

$$\begin{aligned} n(t) &= (1-\alpha)^{\frac{1}{\alpha}} w(t)^{-\frac{1}{\alpha}}\left[\int_0^{M(t)}[x(i,t)]^\alpha di\right]^{\frac{1}{\alpha}} \\ x(i,t) &= \alpha^{\frac{1}{1-\alpha}}[p(i,t)]^{-\frac{1}{1-\alpha}} n(t) \end{aligned}$$

Solution: 3. Solve the IGP i problem

$$\pi(i,t) = \max_{x(i,t),p(i,t)} \; p(i,t)\cdot x(i,t) - x(i,t)$$

IGP i internalizes the FGP demand for $x(i,t)$

$$\pi(i,t) = \max_{x(i,t)} \; p(x(i,t))\cdot x(i,t) - x(i,t)$$

-you can either replace $x(i,t)$ or $p(i,t)$ from the FGP demand-

From FGP FONC $\{x(i,t)\}$: $p(i,t) = \alpha n(t)^{1-\alpha} x(i,t)^{\alpha-1} := p(x(i,t))$

$$\pi(i,t) = \max_{x(i,t)} \; \alpha n(t)^{1-\alpha} x(i,t)^{\alpha-1}\cdot x(i,t) - x(i,t)$$
$$\pi(i,t) = \max_{x(i,t)} \; \alpha n(t)^{1-\alpha} x(i,t)^{\alpha} - x(i,t)$$

FONC

$$x(i,t) \quad : \quad \alpha^2 n(t)^{1-\alpha} x(i,t)^{\alpha-1} - 1 = 0$$

In CE $n_t^* = 1$, solving for $x(i,t)$:

$$x(i,t) = \alpha^{\frac{2}{1-\alpha}}$$

In demand:

$$p(i,t) = \alpha\left(\alpha^{\frac{2}{1-\alpha}}\right)^{\alpha-1} = \frac{1}{\alpha}$$

Solution: 3. Solve the IGP i problem

In CE $n_t^* = 1$, solving for $x(i,t)$:

$$x(i,t) = \alpha^{\frac{2}{1-\alpha}}$$

In demand:

$$p(i,t) = \alpha\left(\alpha^{\frac{2}{1-\alpha}}\right)^{\alpha-1} = \frac{1}{\alpha}$$

Profits:

$$\begin{aligned}
\pi(i,t) = p(i,t)\cdot x(i,t) - x(i,t) &= x(i,t)(p(i,t)-1) \\
&= \alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right) \\
&= \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha)
\end{aligned}$$

Note:

$$\begin{aligned}
x(i,t) = \alpha^{\frac{2}{1-\alpha}} &= \bar{x} \\
p(i,t) = \frac{1}{\alpha} &= \bar{p} \\
\pi(i,t) = \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha) &= \bar{\pi}
\end{aligned}$$

Solution: 4. Derive the BGP growth rate of consumption and GDP-per-capita.

(1) Using EE, find an expression for g_c and a condition on r_{t+1} in the BGP

- From EE:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta(1 + r_{t+1})$$

Note that if there exists a BGP for consumption

$$g_c = \frac{c_{t+1}}{c_t} = \left[\beta(1 + r_{t+1})\right]^{\frac{1}{\gamma}}$$

Then r_{t+1} must be constant

Solution: 4. Derive the BGP growth rate of consumption and GDP-per-capita.

(2) From $V(t)$, and free entry of new ideas, find $\bar{r}$

$$V(t) = \sum_{j=0}^{\infty} \frac{\pi_{i,t+j}}{(1 + r_{t+j})^j}$$

- Replacing $\bar{\pi}, \bar{r}$:

$$\begin{aligned} V(t) &= \sum_{j=0}^{\infty} \frac{\bar{\pi}}{(1+\bar{r})^j} \\ &= \bar{\pi} \sum_{j=0}^{\infty} \frac{1}{(1+\bar{r})^j} \\ &= \bar{\pi} \left(\frac{1}{1 - \frac{1}{1+\bar{r}}} \right) \\ &= \bar{\pi} \left(\frac{1+\bar{r}}{\bar{r}} \right) = \bar{V} \end{aligned}$$

- Free entry implies: $V(t+1)\eta Z_t - (1+r)Z_t = 0 \implies \bar{V}\eta = 1 + \bar{r}$

Solution: 4. Derive the BGP growth rate of consumption and GDP-per-capita.

(2) From $V(t)$, and free entry of new ideas, find $\bar{r}$

- Replacing $\bar{V} = \bar{\pi}\left(\frac{1+\bar{r}}{\bar{r}}\right)$

$$\begin{aligned} \bar{V}\eta &= 1+\bar{r} \\ \bar{\pi}\left(\frac{1+\bar{r}}{\bar{r}}\right)\eta &= 1+\bar{r} \\ \bar{\pi}\left(\frac{1}{\bar{r}}\right)\eta &= 1 \\ \bar{\pi}\eta &= \bar{r} \end{aligned}$$

- In the BGP, the interest rate will be

$$\begin{aligned} \bar{r} &= \bar{\pi}\eta \\ &= \alpha^{\frac{\alpha+1}{1-\alpha}}(1-\alpha)\eta \end{aligned}$$

Solution: 4. Derive the BGP growth rate of C and Y

(3) From $\triangle M(t+1) \equiv M(t+1) - M(t) = \eta \cdot Z(t)$ find a relationship btw g_M and g_Z in BGP

- It must be that $g_M = g_Z$

(4) Derive g_Y and find its relationship with g_M

Note

$$
\begin{aligned}
Y(t) &= n(t)^{1-\alpha} \int_0^{M(t)} \bar{x}^{\alpha} di \\
&= n(t)^{1-\alpha} \bar{x}^{\alpha} M(t) \\
&= \bar{x}^{\alpha} M(t)
\end{aligned}
$$

- $g_Y = \frac{Y(t+1)}{Y(t)} = \frac{\bar{x}^{\alpha} M(t+1)}{\bar{x}^{\alpha} M(t)} = g_M$

$\longrightarrow$ In this 5 steps we found an expression for g_c, proved $g_M = g_Z$ and $g_Y = g_M$ (so $g_Y = g_M = g_Z$)

(5) Finally, use good market clearing to find a relationship btw g_Y and g_c

Good market clears: $Y_t = C_t + X_t + Z_t$

Solution: 4. Derive the BGP growth rate of C and Y

(5) Use good market clearing to find a relationship btw g_Y and g_c

- Good market clears: $Y_t = C_t + X_t + Z_t$

$$\begin{aligned} Y_t &= C_t + X_t + Z_t && (1) \\ &= C_t + \int_0^{M(t)} x(i,t)di + Z_t && (2) \\ &= C_t + \bar{x}M(t) + Z_t && (3) \end{aligned}$$

- Reordering

$$\begin{aligned} C_t &= Y_t - \bar{x}M(t) - Z_t \\ C_{\underline{0}} g_c^t &= Y_{\underline{0}} g_Y^t - \bar{x}M(\underline{0}) g_M^t - Z_{\underline{0}} g_Z^t \end{aligned}$$

- Recall $g_Z = g_M = g_Y$

$$\begin{aligned} C_{\underline{0}} g_c^t &= Y_{\underline{0}} g_Y^t - \bar{x}M(\underline{0}) g_Y^t - Z_{\underline{0}} g_Y^t \\ &= g_Y^t (Y_{\underline{0}} - \bar{x}M(\underline{0}) - Z_{\underline{0}}) \end{aligned}$$

- From (3) note: $Y_{\underline{0}} = C_{\underline{0}} + \bar{x}M(\underline{0}) + Z_{\underline{0}}$
  $\rightarrow C_{\underline{0}} = Y_{\underline{0}} - \bar{x}M(\underline{0}) - Z_{\underline{0}}$
- Concluding $g_Y = g_C$

Practice Question: Final 2023 - Expanding Varieties

1. Characterize the household's optimal choices of consumption, capital, and labor.

(Added) Write the profit maximization problem of the FGP.

(Added) Write the profit maximization problem of the IGP.

2. Derive the FGP's demand function for labor n and intermediates goods x.
3. Solve the intermediate good producers problem and show that in equilibrium the owner of each variety makes period profits.
$$\pi^* = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$$
4. Derive the BGP growth rate of consumption and GDP-per-capita.
5. Formulate the Benevolent Social Planner's (BSP) problem.
6. Prove that the decentralized equilibrium is inefficient. Provide explanation for why the BSP allocation differs from the decentralized market outcome.

Solution: 5. Benevolent Social Planner's Problem

BSP maximizes lifetime discounted utility s.t. resource constraint and production technologies

$$\max_{\{c_t, Z_t, n_t, \{x(i,t)\}_{i=0}^{M_t}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$Y_t = n_t^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di \tag{1}$$

$$M_{t+1} = \eta Z_t + M_t \tag{2}$$

$$Y_t = C_t + X_t + Z_t \tag{3}$$

Solution: 5. Benevolent Social Planner's Problem

[Just for exposition, useful for HW3] In HW3, Problem 2 language:

We know efficiency dictates cost minimization in every productive technology

$$\min_{\{x(i,t)\}_{i=0}^{M(t)},M(t)} \int_0^{M(t)} x(i,t)di$$

$$\text{s.t.} \quad Y(t) \leq n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di$$

$$\mathcal{L} = \int_0^{M(t)} x(i,t)di + \lambda(Y(t) - n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di)$$

FONC

$$\begin{aligned}
x(i,t) \quad &: \quad 1 - \lambda\alpha x(i,t)^{\alpha-1} = 0 \\
&: \quad 1 = \lambda\alpha x(i,t)^{\alpha-1} \\
&: \quad 1 = \lambda\alpha x(M(t),t)^{\alpha-1} \qquad \text{for } i = M(t) \\
M(t) \quad &: \quad x(M(t),t) - \lambda(n(t)^{1-\alpha} x(M(t),t)^\alpha) = 0 \\
&: \quad x(M(t),t) = \lambda(n(t)^{1-\alpha} x(M(t),t)^\alpha)
\end{aligned}$$

*To learn how to solve for the FONC w.r. to $M(t)$ see Leibniz Rule in following slide.

Useful Tool: Leibniz Rule

Leibniz rule is useful for finding $F'(M(t)) = \frac{\partial F(M(t))}{\partial M(t)}$ when/where:

- $F(M(t)) := n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di$
- $F(M(t)) := \int_0^{M(t)} [x(i,t)] di$

Leibniz Rule

Say you have an integral of the form

$$F(x) = \int_{a(x)}^{b(x)} g(x,i)\, di$$

Then the derivative on x is:

$$F'(x) = g(x, b(x)) \cdot b'(x) - g(x, a(x)) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial g(x,i)}{\partial x}\, di$$

In this type of model:

$$F(M_t) = \int_0^{M_t} g(i)\, di$$

$$F'(M_t) = g(M_t) \cdot 1$$

Solution: 5. Benevolent Social Planner's Problem

In lectures' language:

We know efficiency dictates cost minimization in every productive technology, think of the BSP problem of choosing $\{x(i,t)\}_{i=0}^{M_t}$ keeping M_t fixed.

$$\max_{\{x(i,t)\}_{i=0}^{M_t}} n(t)^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di - \int_0^{M(t)} x(i,t) di$$

with FONC:

$$x(i,t) \quad : \quad \alpha n^{1-\alpha} x(i,t)^{\alpha-1} - 1 = 0 \implies x(i,t) = \alpha^{\frac{1}{1-\alpha}} n$$

Now we know

X_t

$$X_t = \int_0^{M(t)} [x(i,t)] di = \int_0^{M(t)} [\alpha^{\frac{1}{1-\alpha}} n] di = \alpha^{\frac{1}{1-\alpha}} n M(t)$$

Y_t

$$Y_t = n^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di = n^{1-\alpha} \int_0^{M(t)} [\alpha^{\frac{1}{1-\alpha}} n]^\alpha di = n\alpha^{\frac{\alpha}{1-\alpha}} M(t)$$

Solution: 5. Benevolent Social Planner's Problem

Now we know

X_t

$$X_t = \int_0^{M(t)} [x(i,t)]di = \int_0^{M(t)} [\alpha^{\frac{1}{1-\alpha}} n]di = \alpha^{\frac{1}{1-\alpha}} nM(t)$$

Y_t

$$Y_t = n^{1-\alpha} \int_0^{M(t)} [x(i,t)]^{\alpha} di = n^{1-\alpha} \int_0^{M(t)} [\alpha^{\frac{1}{1-\alpha}} n]^{\alpha} di = \alpha^{\frac{\alpha}{1-\alpha}} nM(t)$$

- Use this to restate resource constraint (3) $Y_t = C_t + X_t + Z_t$

$$C_t + Z_t = Y_t - X_t = \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t$$

- And replace into (2) $M_{t+1} - M_t = \eta Z_t$

$$M_{t+1} - M_t = \eta(Y_t - X_t - C_t) = \eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t - \eta C_t$$

$$M_{t+1} = \eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) nM_t + M_t - \eta C_t$$

$$M_{t+1} = \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t$$

Solution: 5. Benevolent Social Planner's Problem

Original BSP:

$$\max_{\{c_t, Z_t, n_t, \{x(i,t)\}_{i=0}^{M_t}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$Y_t = n_t^{1-\alpha} \int_0^{M(t)} [x(i,t)]^\alpha di \tag{1}$$

$$M_{t+1} = \eta Z_t + M_t \tag{2}$$

$$Y_t = C_t + X_t + Z_t \tag{3}$$

Now:

$$\max_{\{c_t, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$M_{t+1} = \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t \tag{4}$$

Solution: 5. Benevolent Social Planner's Problem

$$\max_{\{c_t, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.

$$M_{t+1} = \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \Lambda_t \left(\left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] M_t - \eta C_t - M_{t+1} \right) \right\}$$

FONC:

$$\begin{aligned} C_t &: \quad C_t^{-\gamma} = \Lambda_t \eta \\ M_{t+1} &: \quad \Lambda_t = \Lambda_{t+1} \beta \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] \end{aligned}$$

Euler Equation:

$$g_c^{sp} \equiv \frac{C_{t+1}}{C_t} = \left(\beta \left[\eta \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) n + 1\right] \right)^{\frac{1}{\gamma}}$$

Practice Question: Final 2023 - Expanding Varieties

1. Characterize the household's optimal choices of consumption, capital, and labor.

(Added) Write the profit maximization problem of the FGP.

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2. Derive the FGP's demand function for labor n and intermediates goods x.
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6. Prove that the decentralized equilibrium is inefficient. Provide explanation for why the BSP allocation differs from the decentralized market outcome.

Solution: 6. Prove decentralized equilibrium is inefficient

We have arrived to:

- $g_c^{sp} > g_c$:

$$\begin{aligned} g_c^{sp} &= \left(\beta \left[\eta \left[\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right] + 1\right]\right)^{\frac{1}{\gamma}} \\ &> \left[\beta\left(1 + \eta \cdot (\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}) \cdot \alpha^{\frac{1}{1-\alpha}}\right)\right]^{\frac{1}{\gamma}} = g_c \end{aligned}$$

- $\bar{x}^{sp} > \bar{x}$:

$$\begin{aligned} \bar{x}^{sp} &= \alpha^{\frac{1}{1-\alpha}} \\ &> \alpha^{\frac{2}{1-\alpha}} = \bar{x} \end{aligned}$$

- In CE, each IGP internalizes the demand they face, which implies they produce smaller amount than efficient, at a higher price ($P > MC$).
- Because profits in the R&D sector depend on $\bar{x}$, less varieties are produced in equilibrium which leads to lower growth.
- Alternative interpretation: BSP values innovation more because each intermediate good is used more intensively.

Extra Slides on Expanding Varieties

- Even though it is a growth model, this practice problem was also about comparing the Competitive Equilibrium and Benevolent Social Planner allocation.

- You might wonder what is the definition of the CE here.

- The next two slides provide the definition of the Sequence-of-Markets Equilibrium.

The Household Problem

The household problem in this practice question was:

$$\max_{c_t, k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq w_t n_t + (1 + r_t)k_t + \pi_t$$

$$0 \leq n_t \leq 1$$

- It might be weird to see capital here, when (1) households do not typically hold capital directly and (2) no agent (firm) appears to be demanding physical capital in this model.
- To help intuition, you should think of k_{t+1} representing the amount of savings households choose to carry into the next period.
  Then, if at time t households hold k_t units of savings (which they chose the previous period), they will receive $(1 + r_t)k_t$ units of the consumption good.
- This is what we typically see in the consumption-saving problem. We can state the same identical problem calling the "amount of savings" a_t (for assets).

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t n_t + (1 + r_t)a_t + \pi_t$$

SoM Equilibrium

A sequence of market equilibrium in this economy consists of paths for prices $\{w_t, r_t, \{p(i,t)\}_{i=0}^{M_t}\}_{t=0}^{\infty}$ and quantities $\{c_t, n_t, k_{t+1}, \{x(i,t)\}_{i=0}^{M_t}, M_{t+1}, Y_t, Z_t\}_{t=0}^{\infty}$ such that:

1. Households maximize utility subject to their budget constraint:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta(1 + r_{t+1})$$

$$n_t = 1$$

2. The final goods producer maximizes profits:

$$n(t) = (1-\alpha)^{1/\alpha} w(t)^{-1/\alpha} \left(\int_0^{M(t)} [x(i,t)]^{\alpha} di\right)^{1/\alpha}$$

$$x(i,t) = \alpha^{1/(1-\alpha)} [p(i,t)]^{-1/(1-\alpha)} n(t)$$

3. Each intermediate goods producer i maximizes profits:

$$x(i,t) = \alpha^{2/(1-\alpha)}$$

$$p(i,t) = 1/\alpha$$

4. Free entry in the R&D sector ensures zero profits: $V_{t+1}(M_{t+1} - M_t) = (1 + r_t)$
5. The number of varieties evolves according to: $M_{t+1} - M_t = \eta Z_t$

SoM Equilibrium (Continuation)

Market Clearing Conditions:

1. Labor:
$$\underbrace{n_t}_{n_t^D} = \underbrace{1}_{n_t^S}$$
2. Capital:
$$\underbrace{Z_t}_{k_t^D} = \underbrace{k_t}_{k_t^S}$$
3. Goods:
$$Y_t = C_t + Z_t + \int_0^{M_t} x(i,t)di$$