Econ 210A
Discussion Section N°4

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Housekeeping

- Midterm: Thursday, Nov 6
- Homework 2: This Saturday, Oct 25 via Gradescope
- Discussions
    - Today: Growth (Lectures 7, 6, 9)
    - Next Thursday, Oct 30: Midterm review

Plan for today

Overview Growth Models

Existence and Uniqueness of Balanced Growth Path

Practice Question: Balanced Growth Path with CES Production Technology (Midterm 2023)

Overview Growth Models

Two relevant objects (Lecture 7)

rr_k $\quad rr_{k,t} = \frac{MRK_{t+\mathbf{1}} + P_{k,t+\mathbf{1}}(1-\delta)}{P_{k,t}} - 1$

g_k $\quad g_k = \frac{k_{t+\mathbf{1}}}{k_t}$

If $P_{k,t} = P_{k,t+1} = 1$

$rr_{k,t} = \frac{MPK_{t+1} + (1-\delta)}{1} - 1 = MPK_{t+1} - 1$

$= f_k'(k) - 1$

by neoclassical assumption: $\lim_{k\to\infty} f_k(k) = 0 \Rightarrow \lim_{k\to\infty} rr_{k,t} = -\delta$

<u>Neoclassical Growth Model</u>:

rr_k $\quad rr_k = f_k(k) - \delta$, if $k \to \infty$, $rr_k \to -$, so capital will not grow forever, reaching a ss $k_{t+1} = k_t$

g_k $\quad g_k = 1$

$\rightarrow$ Conclusion: Neoclassical growth model admits growth (only) when transitioning towards steady state.

Overview of Growth Models

Note "x grows at rate μ" ?

$\frac{x_{t+1}}{x_t} = \exp(\mu)$ / log

$\log\left(\frac{x_{t+1}}{x_t}\right) = \mu$

$\log x_{t+1} - \log x_t = \mu$

Solow I
Neutral TFP growth

$$f(k_t) = z_t^{1-\alpha} k_t^{\alpha}$$

$$z_{t+1} = z_t \exp(\mu)$$

rr_k As $k \to \infty$: $rr_k > 0$

g_k $g_k = g_z = \exp(\mu)$

Solow II
Embodied TFP growth

$$f(k_t) = k_t^{\alpha}$$

$$k_{t+1} = q_{t+1} i_t + (1-\delta) k_t$$
$$q_{t+1} = q_t \exp(\mu)$$

rr_k As $k \to \infty$: $rr_k > 0$

g_k $g_k = \exp\left(\frac{\mu}{1-\alpha}\right)$

AK
No decreasing returns to capital

$$f(k_t) = A k_t$$

rr_k As $k \to \infty$: $rr_k = A - \delta > 0$

g_k $g_k = \left[\beta(A + 1 - \delta)\right]^{1/\gamma}$

Existence and Uniqueness of Balanced Growth Path (BGP)

Fixing concepts:

- **SS:** In a steady state, the economy stops growing because it reaches a constant level of capital.
- **BGP:** Economy grows at a steady state rate over time.

Relationship btw SS and BGP:

- In the Neoclassical Model, we proved existence of a unique steady state capital k^{ss}

$$k^{ss} = f'^{-1}\left(\frac{1}{\beta} - (1-\delta)\right)$$

- When trying to characterize a solution for Solow I, we noticed exogenous growth of z_t made $\Gamma(k)$ unbounded (violating compactness)
- By redefining all variables $\tilde{x}_t = \frac{x_t}{z_t}$ we get a transformed model that satisfies S&L assumptions
  - So the transformed model has a unique ss defined by

$$\tilde{k}^{ss} = f'^{-1}(\tilde{\beta}^{-1} - (1-\delta)), \qquad \tilde{\beta} = \beta e^{\mu(1-\gamma)}$$

  - $\tilde{k}^{ss} \implies \frac{k_{t+1}}{z_{t+1}} = \frac{k_t}{z_t} \implies \frac{k_{t+1}}{k_t} = \frac{z_{t+1}}{z_t} = exp(\mu)$
  - A proof of existence and uniqueness of a BGP.

Extra : What is the value of $\lim_{t\to\infty} rr_{k,t}$ in Solow I ?

- $rr_{\tilde{k},t} = f_{\tilde{k}_{t+1}}(\tilde{k}_{t+1}) - \delta = \alpha \tilde{k}_{t+1}^{\alpha-1} - \delta$; $\lim_{t\to\infty} rr_{k,t} = \alpha \tilde{k}_{ss}^{\alpha-1} - \delta$

$\rightarrow \tilde{k}_{ss} = (f')^{-1}\left(\frac{1}{\tilde{\beta}} - (1-\delta)\right)$; $\tilde{\beta} = \beta e^{\mu(1-\gamma)}$

$$\underbrace{f'(\tilde{k}_{ss})} = \frac{1}{\tilde{\beta}} - (1-\delta)$$

$$\alpha \tilde{k}_{ss}^{\alpha-1} = \frac{1}{\tilde{\beta}} - (1-\delta) \qquad / \ (\quad)^{\frac{1}{\alpha-1}}$$

$$\rightarrow \tilde{k}_{ss} = \left[\frac{1}{\alpha}\left(\frac{1}{\tilde{\beta}} - (1-\delta)\right)\right]^{\frac{1}{\alpha-1}} \ //$$

- $rr_{\tilde{k},t\to\infty} = \alpha \tilde{k}_{ss}^{\alpha-1} - \delta$

$$= \alpha\left[\frac{1}{\alpha}\left(\frac{1}{\tilde{\beta}} - (1-\delta)\right)\right]^{\frac{\alpha-1}{\alpha-1}} - \delta = \frac{1}{\tilde{\beta}} - 1 \Rightarrow \left[\lim_{t\to\infty} rr_{k,t} = \frac{1}{\tilde{\beta}} - 1 > 0 \quad \square\right]$$

Existence and Uniqueness of Balanced Growth Path

For Solow II

1. Transform to Solow I: $\hat{k}_t = \frac{k_t}{q_t}$
   This transformation gives you Solow I with
$$\left[z_t = q_t^{\frac{\alpha}{1-\alpha}} \right]$$
Explanation in next slide
$$z_{t+1} = exp\left(\frac{\alpha}{1-\alpha}\mu\right) z_t$$
2. Transform to Neoclassical: $\tilde{k}_t = \frac{\hat{k}_t}{z_t}$, where $z_t = (q_t z_t)^{\alpha}$

With this we get a transformed model for which the Neoclassical Model results hold, so $\tilde{k}^{ss}$ exists and is unique

- $\tilde{k}^{ss} \implies \frac{\hat{k}_{t+1}}{\hat{k}_t} = \frac{z_{t+1}}{z_t} = \left(\frac{q_{t+1}}{q_t}\right)^{\frac{\alpha}{1-\alpha}} = exp(\mu)^{\frac{\alpha}{1-\alpha}} = exp\left(\frac{\alpha}{1-\alpha}\mu\right)$
- We have found $g_{\hat{k}} = exp\left(\frac{\alpha}{1-\alpha}\mu\right)$
- For finding g_k note $\frac{\hat{k}_{t+1}}{\hat{k}_t} = \frac{k_{t+1}}{k_t}\frac{q_t}{q_{t+1}} = g_k \frac{1}{exp(\mu)} = exp\left(\frac{\alpha}{1-\alpha}\mu\right)$
- From the last equality we can solve for g_k
$$g_k = exp\left(\frac{\mu}{1-\alpha}\right)$$
- A proof of existence and uniqueness of a BGP.

Extra : Why $z_t = q_t^{\frac{\alpha}{1-\alpha}}$

First, note that in an EXOGENOUS GROWTH MODEL, I would always divide by the process that is EXOGENOUSLY growing over time. Here, that is q_t. However, what is left is discovering the relationship between q_t and the "z_t" of the solow I model.

we want to make SOLOW II look like SOLOW I

	SOLOW II	SOLOW I
production function :	$f(k_t) = k_t^{\alpha}$	$f(k_t) = z_t^{1-\alpha} k_t^{\alpha}$
law of motion for k :	$k_{t+1} = q_{t+1} i_t + (1-\delta) k_t$	$k_{t+1} = i_t + (1-\delta) k_t$
exogenous process :	$q_{t+1} = q_t \exp(\mu)$	$z_{t+1} = z_t \exp(\mu)$
aggregate resource constraint :	$c_t + i_t = f(k_t)$	$c_t + i_t = f(k_t)$

→ what is the same ? aggregate resource constraint , exogenous process

→ what is different ?

PRODUCTION FUNCTION

lets see if we can transform it.

Note $f(k_t) = k_t^{\alpha} = \left(\frac{k_t}{q_t}\right)^{\alpha} q_t^{\alpha}$

$\Rightarrow \quad f(\hat{k}_t) = \hat{k}_t^{\alpha} \; q_t^{\alpha}$

ok nice but in solow I $f(k_t) = z_t^{1-\alpha} k_t^{\alpha}$

here $f(\hat{k}_t) = q_t^{\alpha} \hat{k}_t^{\alpha}$

then it must be that $z_t^{1-\alpha} = q_t^{\alpha}$ □

Note that this is the same way we did in in the practice problem. so always try this ☺

LAW OF MOTION FOR K

make it similar dividing by q_{t+1}, $\hat{x} = \frac{x}{q}$

$$\hat{k}_{t+1} = i_t + (1-\delta) \frac{k_t}{q_t} \frac{q_t}{q_{t+1}}$$

$$\hat{k}_{t+1} = i_t + (1-\delta) \hat{k}_t \exp(-\mu) \quad ✓$$

Practice Question: Midterm 2023

Problem 3: Balanced Growth with a CES Production Technology.
Consider the social planner's problem maximizing,

$$\max_{c_t, k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

subject to the aggregate resource constraint given by

$$c_t + k_{t+1} \leq F(k_t, n_t) + (1-\delta)k_t$$

where the aggregate production technology is in the CES family given by,

$$F(k_t, n_t) = A[\,\alpha k_t^{\rho} + (1-\alpha) n_t^{\rho}\,]^{\frac{1}{\rho}}$$

where $0 < \alpha < 1$, A is aggregate TFP, and $\rho > 0$ is a function of the elasticity of substitution σ between capital k_t and labor n_t, such that

$$\rho = 1 - \frac{1}{\sigma} \iff \sigma = \frac{1}{1-\rho}$$

where $\sigma < 1$ when inputs are gross complements and $\sigma > 1$ when they are gross substitutes.

Practice Question: Balanced Growth with CES

1. Demonstrate that the CES production technology is more general than the Cobb-Douglas form by showing that it is nested as a special case when $\sigma \to 1$ such that

$$\lim_{\sigma \to 1} F(k_t, n_t) \to Ak_t^{\alpha} n_t^{1-\alpha}$$

Hint: take the log of $F(k, n)$ to use L'Hopital's rule and the fact that if $y(x) = a^{g(x)}$ for some constant a, then the derivative $y'(x) = g'(x)a^{g(x)} \ln(a)$

2. Prove that there exists a unique balanced growth path (BGP) when $\sigma \to 1$ and TFP A_t grows exogenously at rate $\mu > 0$ such that,

Note: " A_t grows exogenously at rate μ "

means $\frac{A_{t+1}}{A_t} = \exp(\mu)$ why not $= \mu$?

$$A_{t+1} = \exp(\mu) A_t$$

Because we typically think in log growth!

$\log A_{t+1} - \log A_t = \mu$

Derive the growth rate of GDP-per-capita along the economy's BGP.

3. Is sustained economic growth possible *without* exogenous increases in TFP A (i.e. $\mu = 0$) when $\sigma \leq 1$? Explain your answer.

4. Is sustained economic growth possible *without* exogenous increases in TFP A (i.e. $\mu = 0$) when $\sigma > 1$? Explain your answer.

Practice Question: BGP with CES

2. Prove that there exists a unique balanced growth path (BGP) when $\sigma \to 1$ and TFP A_t grows exogenously at rate $\mu > 0$ such that,

$$A_{t+1} = \exp(\mu) A_t$$

Derive the growth rate of GDP-per-capita along the economy's BGP.

Solution: To prove existence and uniqueness of BGP, we will map this model to one we already know and make a similar argument.
First, note that

$$\lim_{\sigma \to 1} F(k_t, n_t) = A k_t^{\alpha} n_t^{1-\alpha}$$

Also note:

1. Because there is no disutility for labor, $n_t = 1$
2. Defining z_t as $A_t = z_t^{1-\alpha}$: $F(k_t) = z_t^{1-\alpha} k_t^{\alpha}$

$$z_{t+1}^{1-\alpha} = exp(\mu) z_t^{1-\alpha}$$
$$\left(\frac{z_{t+1}}{z_t}\right)^{1-\alpha} = exp(\mu)$$
$$\frac{z_{t+1}}{z_t} = exp(\mu)^{\frac{1}{1-\alpha}}$$
$$\frac{z_{t+1}}{z_t} = exp\left(\frac{\mu}{1-\alpha}\right)$$

Practice Question: BGP with CES

2. Prove that there exists a unique balanced growth path (BGP) when $\sigma \to 1$ and TFP A_t grows exogenously at rate $\mu > 0$ such that,

$$A_{t+1} = \exp(\mu) A_t$$

Derive the growth rate of GDP-per-capita along the economy's BGP.

Solution: To prove existence and uniqueness of BGP, we will map this model one we already know and make a similar argument.

- Then our problem is equivalent to Solow I with

$$\frac{z_{t+1}}{z_t} = exp\left(\frac{\mu}{1-\alpha}\right)$$

- We know Solow I has a unique BGP with

$$g_k = g_z = exp\left(\frac{\mu}{1-\alpha}\right)$$

Practice Question: BGP with CES

2. Derive the growth rate of GDP-per-capita along the economy's BGP.

Alternative solution strategy: Given we can prove existence, here is a method to find the BGP.

(1) Set the SP and find the EE to get an expression for g_c. What does it tell you about g_y and g_k?

(2) Use the resource constraint to find an expression for g_k. What can you learn about g_c and g_k?

(3) Using the production function, solve for g_k

Practice Question: BGP with CES

3. Is sustained economic growth possible *without* exogenous increases in TFP A (i.e. $\mu = 0$) when $\sigma \leq 1$? Explain your answer.

Solution:

- Recall $rr_{k,t} = \frac{MPK_{t+\mathbf{1}} + P_{k,t+\mathbf{1}}(\mathbf{1}-\delta)}{P_{k,t}} - 1$
- To check if a production function can sustain growth we can check if $\lim_{k\to\infty} rr_k$ is a positive constant.
- with $P_{k,t+1} = P_{k,t} = 1$:

$$rr_{k,t} = MPK_{t+1} - \delta$$

- So we will focus on $\lim_{k\to\infty} MPK$

$\rightarrow$ Lets get to work: Find an expression for $MPK = \frac{\partial F}{\partial k}$ under the CES production function. Use $n_t = 1$.

Practice Question: BGP with CES

3. Is sustained economic growth possible *without* exogenous increases in TFP A (i.e. $\mu = 0$) when $\sigma \leq 1$? Explain your answer.

Solution:

$$F(k_t, n_t) = A[\alpha k_t^{\rho} + (1-\alpha)n_t^{\rho}]^{\frac{1}{\rho}}$$

$$MPK = A[\alpha k_t^{\rho} + (1-\alpha)]^{\frac{1}{\rho}-1}\alpha k_t^{\rho-1}$$

$$= A[\alpha k_t^{\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}}\alpha(k_t^{\rho-1})^{\frac{1-\rho}{\rho}\frac{\rho}{1-\rho}}$$

$$= A[\alpha k_t^{\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}}\alpha(k_t^{-\rho})^{\frac{1-\rho}{\rho}}$$

$$= A[\alpha + (1-\alpha)k_t^{-\rho}]^{\frac{1-\rho}{\rho}}\alpha$$

- We arrived to

$$MRK = \alpha A[\alpha + (1-\alpha)k_t^{-\rho}]^{\frac{1}{\rho}-1}$$

Practice Question: BGP with CES

3. Is sustained economic growth possible *without* exogenous increases in TFP A (i.e. $\mu = 0$) when $\sigma \leq 1$? Explain your answer.

Solution:

- We arrived to
$$MPK = \alpha A[\alpha + (1-\alpha)k_t^{-\rho}]^{\frac{1}{\rho}-1}$$
- Note: $\sigma \leq 1 \implies \rho = 1 - \frac{1}{\sigma} \leq 0$
- $\lim_{k \to \infty} MRK = 0$

$\rightarrow$ To see this note the expression inside brackets goes to ∞ but the negative exponent explains the 0 limit.

- Concluding: No sustained economic growth.

Practice Question: BGP with CES

4. Is sustained economic growth possible *without* exogenous increases in TFP A (i.e. $\mu = 0$) when $\sigma > 1$? Explain your answer.

Solution:

- We have

$$MPK = \alpha A[\alpha + (1-\alpha)k_t^{-\rho}]^{\frac{1}{\rho}-1}$$

- Note: $\sigma > 1 \implies \rho = 1 - \frac{1}{\sigma} > 0$

- $\lim_{k\to\infty} MRK = A\alpha^{\frac{1}{\rho}}$

$\longrightarrow$ To see this note $k_t^{-\rho} \to 0$, so the bracket $\to \alpha^{\frac{1}{\rho}-1}$.

- Concluding: $\lim_{k\to\infty} MPK$ is a positive constant, and assuming its larger than δ, the model does support endogenous sustained economic growth.

Cool This is the Ak model with constant A$\alpha^{\frac{1}{\rho}}$

Practice Question: BGP with CES

1. Demonstrate that the CES production technology is more general than the Cobb-Douglas form by showing that it is nested as a special case when $\sigma \to 1$ such that

$$\lim_{\sigma \to 1} F(k_t, n_t) \to A k_t^{\alpha} n_t^{1-\alpha}$$

Hint: take the log of $F(k, n)$ to use L'Hopital's rule and the fact that if $y(x) = a^{g(x)}$ for some constant a, then the derivative $y'(x) = g'(x) a^{g(x)} \ln(a)$

Solution:

- Note $F(k_t, n_t) = exp[log\{F(k_t, n_t)\}]$

  $log\{F(k_t, n_t)\} = logA + \frac{1}{\rho} log[\alpha k_t^{\rho} + (1-\alpha) n_t^{\rho}]$

  $F(k_t, n_t) = exp(logA) exp(\frac{1}{\rho} log[\alpha k_t^{\rho} + (1-\alpha) n_t^{\rho}])$

  $= A \; exp(\frac{1}{\rho} log[\alpha k_t^{\rho} + (1-\alpha) n_t^{\rho}])$

  $\lim_{\rho \to 0} F(k_t, n_t) = A \; exp\Big(\lim_{\rho \to 0} \frac{1}{\rho} log[\alpha k_t^{\rho} + (1-\alpha) n_t^{\rho}] \Big)$

- If we replace by $\rho = 0$ the expression inside *exp* is $\frac{0}{0}$, so we can use L'H for that expression.

CES converges to Cobb-Douglas when $\rho \to 0$

1. **Hint:** take the log of $F(k, n)$ to use L'Hopital's rule and the fact that if $y(x) = a^{g(x)}$ for some constant a, then the derivative $y'(x) = g'(x)a^{g(x)} \ln(a)$

Solution:

- If we replace by $\rho = 0$ the expression inside *exp* is $\frac{0}{0}$, so we can use L'H for that expression.

  $\lim_{\rho \to 0} \frac{log[\alpha k_t^\rho + (1-\alpha)n_t^\rho]}{\rho} = \frac{\lim_{\rho \to 0} \frac{\partial \textbf{num}}{\partial \rho}}{\lim_{\rho \to 0} \frac{\partial \textbf{den}}{\partial \rho}}$

$\to$ $\frac{\partial \text{num}}{\partial \rho} = \frac{1}{\alpha k_t^\rho + (1-\alpha)n_t^\rho}(\alpha k_t^\rho logk_t + (1-\alpha)n_t^\rho logn_t)$

$\to$ $\frac{\partial \text{den}}{\partial \rho} = 1$

  $\lim_{\rho \to 0} \frac{\partial \text{num}}{\partial \rho} = \alpha logk_t + (1-\alpha)logn_t$

- $\lim_{\rho \to 0} \frac{log[\alpha k_t^\rho + (1-\alpha)n_t^\rho]}{\rho} = \alpha logk_t + (1-\alpha)logn_t$

CES converges to Cobb-Douglas when $\rho \to 0$

- We found $\lim_{\rho \to 0} \frac{1}{\rho} log[\alpha k_t^\rho + (1-\alpha) n_t^\rho] = \alpha log k_t + (1-\alpha) log n_t$
- Replacing back into our expression for $\lim_{\rho \to 0} F(k_t, n_t)$:

$$\lim_{\rho \to 0} F(k_t, n_t) = A\ exp\Big(\lim_{\rho \to 0} \frac{1}{\rho} log[\alpha k_t^\rho + (1-\alpha) n_t^\rho] \Big)$$

$$\lim_{\rho \to 0} F(k_t, n_t) = A\ exp(\alpha log k_t + (1-\alpha) log n_t)$$

$$\lim_{\rho \to 0} F(k_t, n_t) = A k_t^\alpha n_t^{1-\alpha}$$

Extra Practice

- Ak model practice in Appendix
  - Midterm 2024 Problem 3: AK Growth with Human Capital.
- For more practice in finding the BGP
  - Midterm 2021 Problem 3: The Neoclassical Growth Model with Endogenous Labor Supply. Solutions: Discussion 5 Carlos Goes.
  - Discussion 6: Endogenous Growth with Human Capital Externalities. By Minki Kim and Carlos Goes.
- Access Carlos' discussions here.

Appendix

Midterm 2024 Problem 3: AK Growth with Human Capital

Midterm 2024 Problem 3: AK Growth with Human Capital

A representative household dynasty maximizes

$$V(h_0) = \max_{c_t, e_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

with the evolution of human capital given by

$$h_{t+1} = \rho h_t + e_t$$

and the resource constraint $c_t + e_t \leq A h_t$.

(A) Derive a condition on A and ρ for growth to be technologically feasible.

Solution 3(a): Technologically feasible growth

$$V(h_0) = \max_{c_t, e_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$h_{t+1} = \rho h_t + e_t$$

$$c_t + e_t \leq A h_t$$

Note that this is the Ak model with

$$i_t = e_t$$

$$k_{t+1} = h_{t+1}$$

$$1 - \delta = \rho$$

Growth is technologically feasible if $A - \delta > 0 \implies A + \rho - 1 > 0$
We can alternatively arrive to a solution noting

$$rr_h = A + \rho - 1$$

And endogenous growth is technologically feasible if

$$lim_{h \to \infty} rr_h = A + \rho - 1 > 0$$

Solution 3(b): Value function is homogeneous in h_0

Substituting in the constraints

$$V(h_0) = \max_{h_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{[(A+\rho)h_t - h_{t+1}]^{1-\gamma}}{1-\gamma}$$

$$\text{st} \quad 0 \leq h_{t+1} \leq (A+\rho)h_t$$

Using $\lambda_t = \frac{h_{t+1}}{h_t}$ the problem is

$$V(h_0) = \max_{\lambda_t} \sum_{t=0}^{\infty} \beta^t h_t^{1-\gamma} \frac{[(A+\rho) - \lambda_t]^{1-\gamma}}{1-\gamma}$$

$$\text{st} \quad 0 \leq \lambda_t \leq (A+\rho)$$

Note

$$h_t = \lambda_{t-1} h_{t-1} = \lambda_{t-1}\lambda_{t-2} h_{t-2} = \prod_{i=0}^{t-1} \lambda_i k_0$$

So

$$V(h_0) = \max_{0 \leq \lambda_t \leq (A+\rho)} \sum_{t=0}^{\infty} \beta^t \prod_{i=0}^{t-1} \lambda_i^{1-\gamma} h_0^{1-\gamma} \frac{[(A+\rho) - \lambda_t]^{1-\gamma}}{1-\gamma}$$

Which we can write as

$$V(h_0) = h_0^{1-\gamma}\omega$$

Problem 3: AK Growth with Human Capital

(Same context, adding the rest of the questions) A representative household dynasty maximizes

$$V(h_0) = \max_{c_t, e_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

s.t.: $h_{t+1} = \rho h_t + e_t$, and $c_t + e_t \leq A h_t$.

(C) Explain computational and analytical advantages of models with homogeneous value functions

(D) Describe sufficient conditions for ω to be expressed recursively, such that

$$\omega = \max_{\lambda} \left\{ \frac{[A + \rho - \lambda]^{1-\gamma}}{1-\gamma} + \beta \lambda^{1-\gamma} \omega \right\} \tag{1}$$

(E) Show that along the optimal growth path, the economy $y_t = A h_t$ grows at rate

$$\frac{y_{t+1}^*}{y_t^*} = [\beta(A + \rho)]^{\frac{1}{\gamma}}$$

(F) Describe how the economy's growth path responds after an unexpected shock (e.g. war, pandemic) which disrupts the accumulation of human capital.

Solution 3(c)-(d)

(C) The fact that the value function is homogeneous in the state means you can scale up the state variable and the value will increase in a predictable scale. Specifically, it is homogeneous of degree $1-\gamma$ in h_0, meaning that if h_0 is scaled by a factor, $V(h_0)$ scales by that factor raised to the power of $1-\gamma$.
The homogeneity of the value function allows for a reduction in the number of unique states that need to be considered when solving the Bellman equation. Instead of needing to solve for every possible level of human capital, we can solve for a representative level and use the homogeneity property to derive values for other levels, which is a major computational advantage.
Finally, it provides analitical tractability.

(D) ω needs to be bounded.

Solution 3(e): Show BGP

Recall Equation (1):

$$\omega = \max_{\lambda} \left\{ \frac{[A + \rho - \lambda]^{1-\gamma}}{1-\gamma} + \beta\lambda^{1-\gamma}\omega \right\}$$

Taking the FONC w.r.t. to λ:

$$\{\lambda\} : -(A + \rho - \lambda^*)^{-\gamma} + \beta(1-\gamma)\lambda^{*-\gamma}\omega^* = 0$$

From which we get an expression for ω^*:

$$\omega^* = \frac{(A + \rho - \lambda^*)^{-\gamma}}{\beta(1-\gamma)\lambda^{*-\gamma}}$$

Replacing back in the optimization problem and setting equal to ω (following equation 1)

$$\omega^* = \frac{[A + \rho - \lambda^*]^{1-\gamma}}{1-\gamma} + \beta\lambda^{*1-\gamma}\omega^*$$

Solution 3(e): Show BGP

Dividing by ω^*: $1 = \frac{1}{\omega^*}\frac{\left[A+\rho-\lambda^*\right]^{1-\gamma}}{(1-\gamma)} + \beta\lambda^{*1-\gamma}$

Replacing $\omega^* = \frac{(A+\rho-\lambda^*)^{-\gamma}}{\beta(1-\gamma)\lambda^{*-\gamma}}$: $1 = \frac{\beta(1-\gamma)\lambda^{*-\gamma}}{(A+\rho-\lambda^*)^{-\gamma}}\frac{\left[A+\rho-\lambda^*\right]^{1-\gamma}}{(1-\gamma)} + \beta\lambda^{*1-\gamma}$

$$1 = \beta\lambda^{*-\gamma}\left[A+\rho-\lambda^*\right] + \beta\lambda^{*1-\gamma}$$

$$\frac{1}{\beta\lambda^{*-\gamma}} = A+\rho-\lambda^* + \lambda^*$$

$$\frac{1}{\beta\lambda^{*-\gamma}} = A+\rho$$

$$\frac{\lambda^{*\gamma}}{\beta} = A+\rho$$

$$\lambda^* = \left[\beta(A+\rho)\right]^{1/\gamma} = \frac{h^*_{t+1}}{h^*_t}$$

Finally, from $y_t = Ah_t$ note $g_y = g_A g_h$ and $g_A = 1$ so $g_y = g_h$

Solution 3(f): Growth path reaction to human capital shocks

(F) Describe how the economy's growth path responds after an unexpected shock (e.g. war, pandemic) which disrupts the accumulation of human capital.

- In the Ak model there is no notion of convergence to steady state or balanced growth path
- If the economy gets shocked and capital is destroyed, the growth rate will be the same
- Note that the expression we derived does not depend on the capital stock h_t