Econ 210A
Discussion Section N°3

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Some Updates

- I've received your homework solutions — they look great!
- Feedback is coming soon; I've just been a bit busy.
- If you're finding the pace tough: slides, HW, and discussions are sufficient. Use the book only if it helps you.
- Following last week's discussion:
  - **Blackwell:** I added an exercise on limits to applying Blackwell's conditions (see Appendix).
  - **Steady state convergence proof:** The assumptions for "monotonic convergence to steady state" are clear in an exercise in the Appendix.

Plan for today

Envelope Condition

From Benevolent Social Planner to Markets

Definition: Sequence of Markets Equilibrium and Recursive Competitive Equilibrium

Practice Question: Inefficient Equilibria

Envelope Theorem

Recall from our last discussion:

General Assumptions	**BSP Neoclassical Model**
A3 X convex	$X = \left[0, \bar{k}\right]$
$\Gamma : X \to X$ non empty, compact, cts	$\Gamma(k_t) = [0, f(k_t) + (1-\delta)k_t]$
A4 $F : X \times X \to \mathbb{R}$ cts, bounded	$F(x_t, x_{t+1}) = u(f(k_t) + (1-\delta)k_t - k_{t+1})$
A5 F strictly increasing in first l arguments	u strictly increasing in k_t
A6 Γ monotone $(x \leq \bar{x} \implies \Gamma(x) \subseteq \Gamma(\bar{x}))$	$\Gamma(k_t)$ monotone in k_t
A7 F strictly concave	u strictly concave
A8 Γ convex	$\Gamma(k_t)$ convex
A9 F cts differentiable on interior of X	u cts differentiable

Result 4 Envelope theorem:

$$v(x) = F(x, g(x)) + \beta v(g(x)) \qquad v(k) = u(f(k) + (1-\delta)k - g(k)) + \beta v(g(k))$$
$$v'(x) = F_1(x, g(x)) \qquad v'(k) = u'(f(k) + (1-\delta)k - g(k))$$

Envelope Condition: Example of Usage of Envelope Theorem

Exercise: Consider the Neoclassical Model

$$v(k_t) = \max_{k_{t+1}} \{u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta v(k_{t+1})\}$$

Using the FONC on k_{t+1} and the Envelope Condition, find the Euler Equation.

(1) FONC $\{k_{t+1}\}$ $\quad u'(\underbrace{f(k_t) + (1-\delta)k_t - k_{t+1}}_{=c_t}) = \beta v'(k_{t+1})$

(2) EC $\quad \dfrac{\partial v(k_t)}{\partial k_t} = u'(c_t)[f'(k_t) + 1 - \delta]$

one period ahead:

$$v'(k_{t+1}) = \frac{\partial v(k_{t+1})}{\partial k_{t+1}} = u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$$

Replace (2) in (1): EE : $\boxed{u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta)}$

Note that you can also use the EC with the Lagrangian

Same problem as last slide but without replacing consumption:

$$V(k) = \max_{c,k'} \left\{ u(c) + \beta V(k') \right\}$$

$$\text{s.t.} \quad c + k' \leq f(k) + (1-\delta) k$$

$\mathcal{L}$: $\quad u(c) + \beta V(k') + \lambda \left(f(k) + (1-\delta) k - c - k' \right)$

FONC $\quad \{c\} \quad u'(c) = \lambda$

$\qquad\quad \{k'\} \quad \beta V'(k') = \lambda$

Envelope Condition $\quad V'(k) = \dfrac{\partial V(k)}{\partial k} = \lambda \left(f'(k) + (1-\delta) \right)$

$$V'(k') = \frac{\partial V(k')}{\partial k'} = \lambda' \left(f'(k') + 1 - \delta \right)$$

EE: $\quad u'(c) = \beta u'(c') \left[f'(k') + 1 - \delta \right]$ $\qquad$ SAME RESULT !!

Neoclassical BSP as a SP vs as a FE problem

Benevolent Social Planner formulation of Neoclassical model now including labor-leisure decision.

Sequential Problem (SP)

$$v(k_0) = \max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\text{s.t.} \quad \begin{aligned} k_{t+1} + c_t &= f(k_t, n_t) + (1-\delta)k_t \\ n_t + l_t &= 1 \\ k_0 & \text{ given} \end{aligned}$$

Solution: $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$

Functional Equation (FE)

$$v(k_t) = \max_{c_t, n_t, k_{t+1}} \left\{ u(c_t, l_t) + \beta v(k_{t+1}) \right\}$$

$$\text{s.t.} \quad \begin{aligned} k_{t+1} + c_t &= f(k_t, n_t) + (1-\delta)k_t \\ n_t + l_t &= 1 \end{aligned}$$

Solution: $v(k)$ and policy functions $c(k)$, $n(k)$, $k'(k)$

Neoclassical Model in Market Formulation

Sequence of Markets Equilibrium

1. Household max lifetime utility st budget constraint

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

s.t. $\quad k_{t+1} + c_t = r_t k_t + w_t n_t + \pi_t + (1-\delta) k_t$

2. Firm max profits

$$\max_{\{k_t^D, n_t^D\}} \pi(k_t^D, n_t^D)$$

3. Markets clear

Solution concept: Sequence of allocations $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$, $\{k_t^D, n_t^D\}_{t=0}^{\infty}$ and prices $\{r_t, w_t\}_{t=0}^{\infty}$ that clear markets.

Recursive Competitive Equilibrium

1. Household max utility given k and K ($G(K) = K'$), st budget constaint

$$v(k, K) = \max_{c,n,k'} u(c, 1-n) + \beta v(k', K')$$

s.t. $\quad k' + c = r(K)k + w(K)n + \pi(K) + (1-\delta)k$

2. Firm max profits

$$\max_{\{K^D, N^D\}} \pi(K^D, N^D)$$

3. Markets clear

4. Expectation for K' is consistent

Solution concept: Value and policy functions of the household $v(k, K)$, $c(k, K)$, $n(k, K)$, $k'(k, K)$, factor demand functions $K^D(K)$, $N^D(K)$ of the firm and price functions $r(K)$, $w(K)$ that clear markets.

Definition: Sequence of Markets (SoM) Equilibrium

A **SoM equilibrium** is a sequence of

1. Prices $\{r_t, w_t\}_{t=0}^{\infty}$
2. Household choices $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$
3. Firm choices $\{n_t^D, k_t^D\}_{t=0}^{\infty}$

such that

a. Given 1: 2 solves household UMP

$$\{c_t, n_t, k_{t+1}\} = argmax \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

st $c_t + k_{t+1} \leq r_t k_t + w_t n_t + (1 - \delta) k_t + \pi_t$ in every $t = 0, 1, \ldots$

b. Given 1: 3 solves the firm profit max problem

$$\{k_t^D, n_t^D\} = argmax \quad \pi(k_t^D, n_t^D)$$

Where $\pi(k_t, n_t) = f(k_t, n_t) - r_t k_t - w_t n_t$ in all periods $t = 0, 1, \ldots$

c. Markets clear

- Goods: $c_t + k_{t+1} = f(k_t, n_t) + (1 - \delta) k_t$
- Capital: $k_t = k_t^D$
- Labor: $n_t = n_t^D$

in all periods $t = 0, 1, \ldots$

Definition: Recursive Competitive Equilibrium (RCE)

A **RCE** is a set of functions of the state variables for

1. Prices $r(K), w(K)$
2. Household value $v(k,K)$, policy functions $c(k,K)$, $n(k,K)$, $k'(k,K)$, and an expectation for evolution of aggregate state $K' = G(K)$
3. Firm demands $N^D(K)$, $K^D(K)$

such that

a. Given 1: 2 solves household UMP

$$v(k,K) = \max_{c,n,k'} u(c, 1-n) + \beta v(k', K')$$

$$\text{s.t.} \quad c + k' - (1-\delta)k \leq r(K)k + w(K)n, \; K' = G(K)$$

b. Given 1: 3 solves the firm profit max problem

$$\{K^D(K), N^D(K)\} = argmax \quad \pi(K,N)$$

Where $\pi(K,N) = f(K,N) - r(K)K - w(K)N$

c. Markets clear
   - Goods: $c + k' = f(K,N) + (1-\delta)k$
   - Capital: $k = K^D$
   - Labor: $n = N^D$

d. Consistency in law of motion of state variable: $k = K \implies K' = G(K)$

Practice Question: Inefficient Equilibria[2]

- We will introduce market failures to illustrate how the competitive equilibrium will not be Pareto Efficient.
- Instantaneous reward function of households depends on pollution (which is an increasing function of total output $P_t = F(Y_t)$).

$$u(c_t, n_t, P_t) = \log c_t + \chi \log(1 - n_t) - D(P_t)$$

  Where D is strictly increasing and differentiable.
- Production function is Cobb Douglas

$$f(N_t, K_t) = AK_t^{\alpha} N_t^{1-\alpha}$$

[2] This question builds from Carlos Goes' Discussion 3, which you can find here.

Practice Question: Inefficient Equilibria

Economy is characterized by

$$u(c_t, n_t, P_t) = \log c_t + \chi \log(1 - n_t) - D(P_t)$$

$$y_t = f(N_t, K_t) = AK_t^{\alpha} N_t^{1-\alpha}$$

Exercise:

1. Competitive equilibrium allocation:
   a. Describe the recursive problem of the household. Use FONC and Envelope to find Euler Equation (EE) and Labor Leisure (LL) Equation.
   b. Describe the problem of the firm and find expressions for labor and capital demand.
   c. Define a RCE in this model.
2. Efficient allocation:
   a. Define the BSP recursive problem.
   b. Using FONC and Envelope on the planner problem, characterize the efficient EE and LL equation.
3. How do the allocations of 1 and 2 differ? Explain.

1.a. Describe the recursive problem of the household.

Solution: Sequential problem of household is household max lifetime utility given prices

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \{\log c_t + \chi \log(1 - n_t) - D(P_t)\}$$

s.t.

$$k_{t+1} + c_t \leq r_t k_t + w_t n_t + (1 - \delta) k_t + \pi_t$$

Recursive problem is household max lifetime utility given k and K

$$v(k, K) = \max_{c,n,k'} \{\log c + \chi \log(1 - n) - D\left[P(K)\right] + \beta v(k', K')\}$$

P[Y(K)]

s.t.

$$c + k' \leq r(K)k + w(K)n + (1 - \delta)k + \pi(K)$$

$$K' = G(K)$$

1.a. Describe the recursive problem of the household.

Solution:

$$v(k,K) = \max_{c,n,k'} \{\log c + \chi \log(1-n) - D\left[P(K)\right] + \beta v(k',K')\}$$

s.t.

hints

$$c + k' \leq r(K)k + w(K)n + (1-\delta)k + \pi(K)$$

(lagrange multiplier for this)

$$K' = G(K)$$

(just replace this one)

Lagrangian:

$$\mathcal{L}(k,K) = \log c + \chi \log(1-n) - D[P(K)] + \beta v(k', G(K)) + \lambda \left(r(K)k + w(K)n + (1-\delta)k + \pi(K) - c - k'\right)$$

FONC:

$$\{c\}: \mathcal{L}_c = \frac{\partial \mathcal{L}}{\partial c} = 0 \quad : \quad \frac{1}{c} = \lambda$$

$$\{k'\}: \mathcal{L}_{k'} = \frac{\partial \mathcal{L}}{\partial k'} = 0 \quad : \quad \beta V_1(k', G(K)) = \lambda$$

$$\{n\}: \mathcal{L}_n = \frac{\partial \mathcal{L}}{\partial n} = 0 \quad : \quad \frac{\chi}{1-n} \cdot -1 + w(K)\lambda \Rightarrow \left[\frac{\chi}{1-n} = \frac{1}{c} w(K)\right] \quad \text{LL} \ \checkmark$$

Envelope:

$$\frac{\partial V(k,K)}{\partial k} = \lambda[r(K) + 1 - \delta] \qquad \frac{\partial V(k',K')}{\partial k'} = \lambda'[r(K') + 1 - \delta] = V_1(k', G(K))$$

$\{c\}: \mathcal{L}_c = \frac{\partial \mathcal{L}}{\partial c} = 0 \quad : \quad \frac{1}{c} = \lambda$

$\{k'\}: \mathcal{L}_{k'} = \frac{\partial \mathcal{L}}{\partial k'} = 0 \quad : \quad \beta V_1(k', G(K)) = \lambda$

$$\frac{\partial V(k,K)}{\partial k} = \lambda [r(K) + 1 - \delta] \qquad \frac{\partial V(k',K')}{\partial k'} = \lambda' [r(K') + 1 - \delta] = V_1(k', G(K))$$

(1) $\{c\}$ and $\{k'\}$: $\quad \frac{1}{c} = \beta V_1(k', G(K))$

(2) EC $\rightsquigarrow$ (1) $\quad \left\{ \frac{1}{c} = \beta \frac{1}{c'} [r(K') + 1 - \delta] \right\}$ EE ✓

1.a. Describe the recursive problem of the household.

this changes w.r.to the BSP problem
now the return of 1 extra unit of k' is
$r(K') + (1-\delta)$ instead of $F_K(K,N) + (1-\delta)$
[however, we can get there using the Firm FONC]

Solution:

EE

the Euler Equation:

Characterizes decision/ trade off btw consumption and savings

$$\underbrace{\frac{1}{c}}_{\text{mg utility c today}} = \underbrace{\beta \frac{1}{c'} [r(K') + (1-\delta)]}_{\text{discounted mg utility of consuming savings tomorrow}}$$

LL

the Labor-Leisure Equation:

Characterizes decision/ trade off btw labor and leisure

$$\underbrace{\frac{\chi}{1-n}}_{\text{mg disutility of work}} = \underbrace{\frac{w(K)}{c}}_{\text{mg utility of consuming mg wage}}$$

same idea here, a BSP would gain $F_N(K,N)$ for each extra hour worked! (Equivalent using firm FONC)

one extra unit of work (say an hour) discounts $\frac{\chi}{1-n}$ units from my utility
[$U_n(c,n)$: mg disutility of labor]

If I work an extra hour, I'll earn $w(K)$ extra resources, which I can spend on consumption and get $[w(K) \cdot \underbrace{\frac{1}{c}}_{\text{mg utility of consumption}}]$ extra "units of utility"

1.b. Describe the problem of the firm and find K^D, N^D

Solution: The firm problem is:

$$\max_{K_d, L_d} \left\{ A K_d^{\alpha} N_d^{1-\alpha} - K_d r(K) - N_d w(K) \right\}$$

FONC:

$\{K_d\}$: $\quad \frac{\partial \Pi}{\partial K} = 0 : \quad \alpha A K_d^{\alpha-1} N_d^{1-\alpha} = r(K)$

$\{N_d\}$: $\quad \frac{\partial \Pi}{\partial N} = 0 : \quad (1-\alpha) A K_d^{\alpha-1} N_d^{-\alpha} = w(K)$

1.c. Define a RCE

A **RCE** is a set of functions of the state variables for

1. Prices $r(K), w(K)$
2. Household value $v(k, K)$, policy functions $c(k, K)$, $n(k, K)$, $k'(k, K)$, and an expectation for evolution of aggregate state $K' = G(K)$
3. Firm demands $N^D(K)$, $K^D(K)$

such that

a. Given 1: 2 solves household UMP

b. Given 1: 3 solves the firm profit max problem

c. Markets clear
   - Goods: $c + k' = AK^{\alpha}N^{1-\alpha} + (1-\delta)k$
   - Capital: $k = K_d$
   - Labor: $n = N_d$

d. Consistency in law of motion of state variable

$$k = K \implies K' = G(K)$$

2.a. Define the BSP recursive problem

A **solution to the recursive planner's problem** is defined as a social value function $V(K)$ and social policy functions $K'(K), N(K), C(K)$ such that the value function and the policy functions maximize lifetime social utility, the allocation is feasible, and resources in society are exhausted, i.e. $C(K) + K'(K) = Y(K) + (1-\delta)K$, with $Y(K) = AK^{\alpha}[N(K)]^{1-\alpha}$.

$$
\begin{aligned}
V(K) \quad &= \quad \max_{C,N,K'} \{\log C + \chi \log(1-N) - D(P) + \beta V(K')\} \\
s.t. \quad & K' + C \leq Y + (1-\delta)K \\
& P = F(Y) \\
& Y = AK^{\alpha} N^{1-\alpha}
\end{aligned}
$$

2.b. Characterize efficient EE and LL using FONC and Envelope

$$V(K) = \max_{C,N,K'} \{\log C + \chi \log(1-N) - D(P) + \beta V(K')\}$$

$$\text{s.t.} \; (1) \; K' + C \leq Y + (1-\delta)K$$

$$(2) \; P = F(Y)$$

$$(3) \; Y = AK^\alpha N^{1-\alpha}$$

<u>Note</u>: I'm going to replace (2)-(3) [equality const.] and include (1) [inequality const.] in Lagrangian

Lagrangian: $\mathcal{L}: \log C + \chi \log(1-N) - D(\underbrace{F(\underbrace{AK^\alpha N^{1-\alpha}}_{Y})}_{P}) + \beta V(K') + \mu[AK^\alpha N^{1-\alpha} + (1-\delta)K - K' - C]$

<u>NOTE</u>: : I replaced (2) & (3) but I can use any of this expressions interchangeably if I want to save notation

<u>Example</u>: $\frac{\partial D(\cdot)}{\partial K} = \frac{\partial D(\cdot)}{\partial P} \cdot \frac{\partial P}{\partial Y} \cdot \frac{\partial Y}{\partial K}$ [chain rule]

$$\text{or} = D'(P) \cdot F'(Y) \cdot f_K(N,K)$$

$$\text{or} = D_P(F(AK^\alpha N^{1-\alpha})) \cdot F_Y(AK^\alpha N^{1-\alpha}) \cdot AK^\alpha N^{1-\alpha}$$

$\Rightarrow$ you choose your notation; just be consistent & you'll be fine

FONC:

$\{C\}$: $\frac{1}{C} = \mu$

$\{N\}$: $\frac{\chi}{1-N} + D'(P)F'(Y)(1-\alpha)AK^\alpha N^{-\alpha} = \mu[(1-\alpha)AK^\alpha N^{-\alpha}]$

$\{K'\}$: $\beta V'(K') = \mu$

Envelope:

$\frac{\partial V(K)}{\partial K} = \frac{\partial \mathcal{L}}{\partial K}$ only considarating DIRECT effect (no reoptimization or INDIRECT effect)

$$V'(K) = -D'(P)F'(Y)\alpha AK^{\alpha-1}N^{1-\alpha} + \mu[\alpha AK^\alpha N^{1-\alpha} + (1-\delta)]$$

$$V'(K') = -D'(P')F'(Y')\alpha AK'^{\alpha-1}N'^{1-\alpha} + \mu'[\alpha AK'^\alpha N'^{1-\alpha} + (1-\delta)]$$

FONC

$\{c\}$: $\quad \dfrac{1}{c} = \mu$

$\{n\}$: $\quad \dfrac{x}{1-N} + D'(P)\,F'(y)\,(1-\alpha)AK^{\alpha}N^{-\alpha} = \mu\left[(1-\alpha)AK^{\alpha}N^{-\alpha}\right]$

$\{k'\}$: $\quad \mu = \beta V'(K')$

Envelope

$\dfrac{\partial V(K)}{\partial K} = \dfrac{\partial \mathcal{L}}{\partial K}$ only considarating DIRECT effect (no reoptimization or INDIRECT effect)

$$V'(K) = -D'(P)F'(y)\alpha AK^{\alpha-1}N^{1-\alpha} + \mu\left[\alpha AK^{\alpha}N^{1-\alpha} + (1-\delta)\right]$$

$$\boxed{V'(K') = -D'(P')F'(y')\alpha AK'^{\alpha-1}N'^{1-\alpha} + \mu'\left[\alpha AK'^{\alpha}N'^{1-\alpha} + (1-\delta)\right]}$$

Replacing $V'(K')$ from Envelope, $\mu' = \frac{1}{c'}$ and $\mu = \frac{1}{c}$ from $\{c\}$ into $\{k'\}$ we get:

$$\boxed{\frac{1}{c} = \beta\left[\frac{1}{c'}\left(\alpha AK'^{\alpha}N'^{1-\alpha} + 1 - \delta\right) - D'(P')F'(y')\alpha AK'^{\alpha-1}N'^{1-\alpha}\right]} \quad \text{EE}$$

Replacing $\mu = \frac{1}{c}$ in $\{n\}$:

$$\boxed{\frac{x}{1-N} + D'(P)F'(y)(1-\alpha)AK^{\alpha}N^{-\alpha} = \frac{1}{c}\left[(1-\alpha)AK^{\alpha}N^{-\alpha}\right]} \quad \text{LL}$$

3. Compare allocations from 1. and 2.

1. RCE

EE (replacing $r(K')$ from K^D)

$$\frac{1}{c} = \beta \frac{1}{c'} [\alpha A (K')^{\alpha-1} (N')^{1-\alpha} + (1-\delta)]$$

LL (replacing $w(K)$ from N^D)

$$\frac{\chi}{1-n} = \frac{(1-\alpha) A K^{\alpha} N^{-\alpha}}{c}$$

2. Planner

EE

$$\frac{1}{C} = \beta \frac{1}{C'} [\alpha A (K')^{\alpha-1} (N')^{1-\alpha} + (1-\delta)] - \beta [D'(P') F'(Y') \alpha A (K')^{\alpha-1} (N')^{1-\alpha}]$$

LL

$$\frac{\chi}{1-N} = \frac{(1-\alpha) A K^{\alpha} N^{-\alpha}}{C} - D'(F(Y)) F'(Y) (1-\alpha) A K^{\alpha} N^{-\alpha}$$

3. EE

RCE

$$\frac{1}{c} = \beta \frac{1}{c'} [\alpha A (K')^{\alpha-1} (N')^{1-\alpha} + (1-\delta)]$$

Planner

$$\frac{1}{C} = \beta \frac{1}{C'} \left[\alpha A (K')^{\alpha-1} (N')^{1-\alpha} + (1-\delta)\right] - \beta \underbrace{D'(P')F'(Y')\alpha A (K')^{\alpha-1} (N')^{1-\alpha}}_{\text{disutility of extra pollution next period}}$$

- Note that they differ by the term in the right, which denotes the discounted disutility of having to experience higher pollution in the next period due to higher production as a function of higher savings.
- The planner internalizes the disutility of extra production that happens through pollution, while households ignore it in the competitive equilibrium.
- In the competitive equilibrium households oversave and overconsume compared to the social optimum.

3. LL

RCE

$$\frac{\chi}{1-n} = \frac{(1-\alpha)AK^{\alpha}N^{-\alpha}}{c}$$

Planner

$$\frac{\chi}{1-N} = \frac{(1-\alpha)AK^{\alpha}N^{-\alpha}}{C} - \underbrace{D'(P)F'(Y)(1-\alpha)AK^{\alpha}N^{-\alpha}}_{\text{disutility of extra pollution}}$$

- Note that they differ by the term in the right, which denotes the disutility of having to experience higher pollution today due to higher production as a function of higher hours worked.
- The planner internalizes the disutility of extra production that happens through pollution, while households ignore it in the competitive equilibrium.
- In the competitive equilibrium households overwork compared to the social optimum.

Appendix

When Blackwell's Conditions do not work

Steady State Proof

Practice Question: Midterm 2022

Pie Eating Problem. Consider the sequence problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\pi_{t+1} = \pi_t - c_t$$

$$0 < c_t \leq \pi_t$$

where $0 < \beta < 1$ and $\pi_0 > 0$ is given. Preferences are in the CRRA family,

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}}{1-\gamma} & \text{if } 0 < \gamma < \infty\,,\ \gamma \neq 1 \\ \ln(c_t) & \text{if } \gamma = 1 \end{cases}$$

1. Express the problem recursively using the FE representation.
2. Define the objects which constitute a solution to the FE problem.
3. Explain why you cannot directly apply Blackwell's Sufficient Conditions to prove that there exists a unique solution.

   Hint: Identify an assumption that is violated with CRRA utility.

Solution: Pie Eating Problem I

1. **Express the problem recursively using the FE representation.**
   The SP is

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\pi_t)$$

$$s.t. \quad \pi_{t+1} = \pi_t - c_t$$

$$0 < c_t \leq \pi_t$$

   The FE is

$$v(\pi) = \max_{\{\pi'\}} \{u(c) + \beta v(\pi')\}$$

$$s.t. \quad \pi' = \pi - c$$

$$0 < c \leq \pi$$

   Which we can also write:

$$v(\pi) = \max_{\{\pi'\}} \{u(\pi - \pi') + \beta v(\pi')\}$$

Solution: Pie Eating Problem II

2. **Define the objects which constitute a solution to the FE problem.**

The solution to this FE problem is a value function $v(\pi)$ which denotes the discounted maximized lifetime utility for an individual with wealth level π at the present period; and policy functions $c(\pi)$ and $\pi'(\pi)$, which denote the optimal choices of consumption and future assets for an individual with wealth level π at the present period.

Solution: Pie Eating Problem III

3. **Explain why you cannot directly apply Blackwell's to prove that there exists a unique solution.**

- You cannot directly apply Blackwell's Sufficient Conditions because $u(c) : \mathbb{R} \to \mathbb{R}$ is not bounded (as c approaches 0, $u(c) \to -\infty$ and when $\gamma \to \infty, u(c) \to \infty$ if $c > 1$).

Steady state k^{ss}

Exercise: Consider the Neoclassical Model

$$v(k_t) = \max_{k_{t+1}} \{u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta v(k_{t+1})\}$$

1. Using the FONC on k_{t+1} and the Envelope Condition, find the Euler Equation. (We did it in the discussion!)
2. Assume a steady state capital (k^{ss}) exists. Find an expression for it. (Try this on your own from EE)

Now the real proof begins:

3. Using the concavity of v and f, prove that the expression you found in 2. is always a steady state.
4. Show that k_t converges monotonically to k^{ss}

Solution: Steady state k^{ss} I

3. Using the concavity of v and f, prove that the expression you found in 2. is always a steady state.

   Claim: If $k^{ss} = f'^{-1}\left(\beta^{-1} - 1 + \delta\right)$, then k^{ss} is a steady state ($g(k^{ss}) = k^{ss}$)

   Proof: Concavity of v implies:

   - If $g(k) \geq k \implies v'(g(k)) \leq v'(k)$
   - If $g(k) \leq k \implies v'(g(k)) \geq v'(k)$
   - Under either:
   $$\left[v'(g(k)) - v'(k)\right][g(k) - k] \leq 0$$

   By substituting

   - $v'(g(k))$ from the FONC $\{k_{t+1}\}$: $u'(c) = \beta v'(g(k)) \implies v'(g(k)) = \beta^{-1}u'(c)$
   - $v'(k)$ from the Envelope Condition: $v'(k) = u'(c)(f'(k) + 1 - \delta)$

   We get

   - $\left[u'(c)\beta^{-1} - u'(c)(f'(k) + 1 - \delta))\right][g(k) - k] \leq 0$
   - Which simplifies to $\left[\beta^{-1} - (f'(k) + 1 - \delta))\right](g(k) - k) \leq 0$
   - Multiplying by -1: $\left[f'(k) + (1 - \delta) - \beta^{-1})\right](g(k) - k) \geq 0$

Solution: Steady state k^{ss} II

We arrived to:

$$\underbrace{\left[f'(k) - \beta^{-1} + (1-\delta)\right]}_{A} \underbrace{[g(k) - k]}_{B} \geq 0 \tag{1}$$

By strict concavity of f:

- If $k > k^{ss}$: $f'(k) < \beta^{-1} - (1-\delta) = f'(k^{ss})$ $(A < 0)$
- If $k < k^{ss}$: $f'(k) > \beta^{-1} - (1-\delta) = f'(k^{ss})$ $(A > 0)$

  Why? Recall k^{ss} is defined by $f'(k^{ss}) = \beta^{-1} - (1-\delta)$

Note that:

- If $g(k) > k \quad (B > 0)$: For $A \times B \geq 0$ we need $A \geq 0$, but if $k > k^{ss}$, $A < 0$ a contradiction with (1)
- If $g(k) < k \quad (B < 0)$: For $A \times B \geq 0$ we need $A \leq 0$, but if $k < k^{ss}$ $A > 0$ a contradiction with (1)
- We showed $g(k) = k = k^{ss}$, which concludes the proof $\square$

Solution: Steady state k^{ss} III

4. Show that k_t converges monotonically to k^{ss}

Claim: $g(k_t)$ converges monotonically to k^{ss}.

$$
\begin{aligned}
k > k^{ss} \quad &\implies \quad k^{ss} < g(k) < k \\
k < k^{ss} \quad &\implies \quad k < g(k) < k^{ss}
\end{aligned}
$$

Proof: Recall equation (1)

$$\underbrace{\left[f'(k) - \beta^{-1} + (1-\delta)\right]}_{A} \underbrace{\left[g(k) - k\right]}_{B} \geq 0$$

Recall at k^{ss}, $f'(k^{ss}) = \beta^{-1} - (1-\delta)$

By strict concavity of f:

- If $k > k^{ss}$: $f'(k) < \beta^{-1} - (1-\delta)$ $(A < 0)$ so it must be that $g(k) - k < 0$ $(B < 0)$ "capital stock shrinks"
- If $k < k^{ss}$: $f'(k) > \beta^{-1} - (1-\delta)$ $(A > 0)$ so it must be that $g(k) - k > 0$ $(B > 0)$ "capital stock grows"

□