

Econ 210A

Discussion Section N°3

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Some Updates

- ▶ I've received your homework solutions — they look great!
- ▶ If you're finding the pace tough: slides, HW, and discussions are sufficient. Use the book only if it helps you.
- ▶ Following last week's discussion:
 - ▶ **Blackwell:** I added an exercise on limits to applying Blackwell's conditions (see Appendix).
 - ▶ **Steady state convergence proof:** The assumptions for “monotonic convergence to steady state” are clear in an exercise in the Appendix.

Plan for today

Envelope Condition

From Benevolent Social Planner to Markets

Definition: Sequence of Markets Equilibrium and Recursive Competitive Equilibrium

Practice Question: Inefficient Equilibria

Envelope Theorem

Recall from our last discussion:

General Assumptions	BSP Neoclassical Model
A3 X convex	$X = [0, \bar{k}]$
$\Gamma : X \rightarrow X$ non empty, compact, cts	$\Gamma(k_t) = [0, f(k_t) + (1 - \delta)k_t]$
A4 $F : X \times X \rightarrow \mathbb{R}$ cts, bounded	$F(x_t, x_{t+1}) = u(f(k_t) + (1 - \delta)k_t - k_{t+1})$
A5 F strictly increasing in first / arguments	u strictly increasing in k_t
A6 Γ monotone ($x \leq \bar{x} \implies \Gamma(x) \subseteq \Gamma(\bar{x})$)	$\Gamma(k_t)$ monotone in k_t
A7 F strictly concave	u strictly concave
A8 Γ convex	$\Gamma(k_t)$ convex
A9 F cts differentiable on interior of X	u cts differentiable

Result 4 Envelope theorem:

$$v(x) = F(x, g(x)) + \beta v(g(x))$$

$$v'(x) = F_1(x, g(x))$$

$$v(k) = u(f(k) + (1 - \delta)k - g(k)) + \beta v(g(k))$$

$$v'(k) = u'(f(k) + (1 - \delta)k - g(k))$$

Envelope Condition: Example of Usage of Envelope Theorem

Exercise: Consider the Neoclassical Model

$$v(k_t) = \max_{k_{t+1}} \{u(f(k_t) + (1 - \delta)k_t - k_{t+1}) + \beta v(k_{t+1})\}$$

Using the FONC on k_{t+1} and the Envelope Condition, find the Euler Equation.

Neoclassical BSP as a SP vs as a FE problem

Benevolent Social Planner formulation of Neoclassical model now including labor-leisure decision.

Sequential Problem (SP)

$$v(k_0) = \max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\begin{aligned} \text{s.t.} \quad & k_{t+1} + c_t = f(k_t, n_t) + (1 - \delta)k_t \\ & n_t + l_t = 1 \\ & k_0 \text{ given} \end{aligned}$$

Solution: $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$

Functional Equation (FE)

$$v(k_t) = \max_{c_t, n_t, k_{t+1}} \left\{ u(c_t, l_t) + \beta v(k_{t+1}) \right\}$$

$$\begin{aligned} \text{s.t.} \quad & k_{t+1} + c_t = f(k_t, n_t) + (1 - \delta)k_t \\ & n_t + l_t = 1 \end{aligned}$$

Solution: $v(k)$ and policy functions $c(k)$, $n(k)$, $k'(k)$

Neoclassical Model in Market Formulation

Sequence of Markets Equilibrium

1. Household max lifetime utility st budget constraint

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

$$\text{s.t.} \quad k_{t+1} + c_t = r_t k_t + w_t n_t + \pi_t + (1 - \delta)k_t$$

2. Firm max profits

$$\max_{\{k_t^D, n_t^D\}} \pi(k_t^D, n_t^D)$$

3. Markets clear

Solution concept: Sequence of allocations $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$, $\{k_t^D, n_t^D\}_{t=0}^{\infty}$ and prices $\{r_t, w_t\}_{t=0}^{\infty}$ that clear markets.

Definition: Recursive Competitive Equilibrium

1. Household max utility given k and K ($G(K) = K'$), st budget constraint

$$v(k, K) = \max_{c, n, k'} u(c, 1 - n) + \beta v(k', K')$$

$$\text{s.t.} \quad k' + c = r(K)k + w(K)n + \pi(K) + (1 - \delta)k$$

2. Firm max profits

$$\max_{\{K^D, N^D\}} \pi(K^D, N^D)$$

3. Markets clear

4. Expectation for K' is consistent

Solution concept: Value and policy functions of the household $v(k, K)$, $c(k, K)$, $n(k, K)$, $k'(k, K)$, factor demand functions $K^D(K)$, $N^D(K)$ of the firm and price functions $r(K)$, $w(K)$ that clear markets.

Sequence of Markets (SoM) Equilibrium

A **SoM equilibrium** is a sequence of

1. Prices $\{r_t, w_t\}_{t=0}^{\infty}$
2. Household choices $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$
3. Firm choices $\{n_t^D, k_t^D\}_{t=0}^{\infty}$

such that

- a. Given 1: 2 solves household UMP

$$\{c_t, n_t, k_{t+1}\} = \underset{t=0}{\operatorname{argmax}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

st $c_t + k_{t+1} \leq r_t k_t + w_t n_t + (1 - \delta)k_t + \pi_t$ in every $t = 0, 1, \dots$

- b. Given 1: 3 solves the firm profit max problem

$$\{k_t^D, n_t^D\} = \underset{}{\operatorname{argmax}} \pi(k_t^D, n_t^D)$$

Where $\pi(k_t, n_t) = f(k_t, n_t) - r_t k_t - w_t n_t$ in all periods $t = 0, 1, \dots$

- c. Markets clear

- ▶ Goods: $c_t + k_{t+1} = f(k_t, n_t) + (1 - \delta)k_t$
- ▶ Capital: $k_t = k_t^D$
- ▶ Labor: $n_t = n_t^D$

in all periods $t = 0, 1, \dots$

Definition: Recursive Competitive Equilibrium (RCE)

A **RCE** is a set of functions of the state variables for

1. Prices $r(K)$, $w(K)$
2. Household value $v(k, K)$, policy functions $c(k, K)$, $n(k, K)$, $k'(k, K)$, and an expectation for evolution of aggregate state $K' = G(K)$
3. Firm demands $N^D(K)$, $K^D(K)$

such that

- a. Given 1: 2 solves household UMP

$$v(k, K) = \max_{c, n, k'} u(c, 1 - n) + \beta v(k', K')$$

$$\text{s.t. } c + k' - (1 - \delta)k \leq r(K)k + w(K)n, K' = G(K)$$

- b. Given 1: 3 solves the firm profit max problem

$$\{K^D(K), N^D(K)\} = \operatorname{argmax} \pi(K, N)$$

Where $\pi(K, N) = f(K, N) - r(K)K - w(K)N$

- c. Markets clear

- ▶ Goods: $c + k' = f(K, N) + (1 - \delta)k$
- ▶ Capital: $k = K^D$
- ▶ Labor: $n = N^D$

- d. Consistency in law of motion of state variable: $k = K \implies K' = G(K)$

Practice Question: Inefficient Equilibria²

- ▶ We will introduce market failures to illustrate how the competitive equilibrium will not be Pareto Efficient.
- ▶ Instantaneous reward function of households depends on pollution (which is an increasing function of total output $P_t = F(Y_t)$).

$$u(c_t, n_t, P_t) = \log c_t + \chi \log(1 - n_t) - D(P_t)$$

Where D is strictly increasing and differentiable.

- ▶ Production function is Cobb Douglas

$$f(N_t, K_t) = AK_t^\alpha N_t^{1-\alpha}$$

²This question builds from Carlos Goes' Discussion 3, which you can find [here](#).

Practice Question: Inefficient Equilibria

Economy is characterized by

$$u(c_t, n_t, P_t) = \log c_t + \chi \log(1 - n_t) - D(P_t)$$

$$f(N_t, K_t) = AK_t^\alpha N_t^{1-\alpha}$$

Exercise:

1. Competitive equilibrium allocation:
 - a. Describe the recursive problem of the household. Use FONC and Envelope to find Euler Equation (EE) and Labor Leisure (LL) Equation.
 - b. Describe the problem of the firm and find expressions for labor and capital demand.
 - c. Define a RCE in this model.
2. Efficient allocation:
 - a. Define the BSP recursive problem.
 - b. Using FONC and Envelope on the planner problem, characterize the efficient EE and LL equation.
3. How do the allocations of 1 and 2 differ? Explain.

1.a. Describe the recursive problem of the household.

Solution: Sequential problem of household is household max lifetime utility given prices

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \{ \log c_t + \chi \log(1 - n_t) - D(P_t) \}$$

s.t.

$$k_{t+1} + c_t \leq r_t k_t + w_t n_t + (1 - \delta)k_t + \pi_t$$

Recursive problem is household max lifetime utility given k and K

$$v(k, K) = \max_{c, n, k'} \{ \log c + \chi \log(1 - n) - D[P(K)] + \beta v(k', K') \}$$

s.t.

$$\begin{aligned} c + k' &\leq r(K)k + w(K)n + (1 - \delta)k + \pi(K) \\ K' &= G(K) \end{aligned}$$

1.a. Describe the recursive problem of the household.

Solution:

$$v(k, K) = \max_{c, n, k'} \{ \log c + \chi \log(1 - n) - D[P(K)] + \beta v(k', K') \}$$

s.t.

$$\begin{aligned} c + k' &\leq r(K)k + w(K)n + (1 - \delta)k + \pi(K) \\ K' &= G(K) \end{aligned}$$

Lagrangian:

FONC:

Envelope:

1.a. Describe the recursive problem of the household.

Solution:

EE

$$\underbrace{\frac{1}{c}}_{\text{mg utility } c \text{ today}} = \underbrace{\beta \frac{1}{c'} [r(K') + (1 - \delta)]}_{\text{discounted mg utility of consuming savings tomorrow}}$$

LL

$$\underbrace{\frac{\chi}{1 - n}}_{\text{mg disutility of work}} = \underbrace{\frac{w(K)}{c}}_{\text{mg utility of consuming mg wage}}$$

1.b. Describe the problem of the firm and find K^D , N^D

Solution: The firm problem is:

$$\max_{K_d, L_d} \{AK_d^\alpha N_d^{1-\alpha} - K_d r(K) - N_d w(K)\}$$

FONC:

$\{K_d\}$:

$\{N_d\}$:

1.c. Define a RCE

A **RCE** is a set of functions of the state variables for

1. Prices $r(K), w(K)$
2. Household value $v(k, K)$, policy functions $c(k, K)$, $n(k, K)$, $k'(k, K)$, and an expectation for evolution of aggregate state $K' = G(K)$
3. Firm demands $N^D(K)$, $K^D(K)$

such that

- a. Given 1: 2 solves household UMP
- b. Given 1: 3 solves the firm profit max problem
- c. Markets clear
 - ▶ Goods: $c + k' = AK^\alpha N^{1-\alpha} + (1 - \delta)k$
 - ▶ Capital: $k = K_d$
 - ▶ Labor: $n = N_d$
- d. Consistency in law of motion of state variable

$$k = K \implies K' = G(K)$$

2.a. Define the BSP recursive problem

A **solution to the recursive planner's problem** is defined as a social value function $V(K)$ and social policy functions $K'(K)$, $N(K)$, $C(K)$ such that the value function and the policy functions maximize lifetime social utility, the allocation is feasible, and resources in society are exhausted, i.e.

$C(K) + K'(K) = Y(K) + (1 - \delta)K$, with $Y(K) = AK^\alpha[N(K)]^{1-\alpha}$.

$$\begin{aligned} V(K) &= \max_{C, N, K'} \{ \log C + \chi \log(1 - N) - D(P) + \beta V(K') \} \\ \text{s.t. } & K' + C \leq Y + (1 - \delta)K \\ & P = F(Y) \\ & Y = AK^\alpha N^{1-\alpha} \end{aligned}$$

2.b. Characterize efficient EE and LL using FONC and Envelope

$$\begin{aligned} V(K) &= \max_{C, N, K'} \{ \log C + \chi \log(1 - N) - D(P) + \beta V(K') \} \\ \text{s.t. } & K' + C \leq Y + (1 - \delta)K \\ & P = F(Y) \\ & Y = AK^\alpha N^{1-\alpha} \end{aligned}$$

Lagrangian:

FONC:

Envelope:

3. Compare allocations from 1. and 2.

1. RCE

EE (replacing $r(K')$ from K^D)

$$\frac{1}{c} = \beta \frac{1}{c'} [\alpha A(K')^{\alpha-1} (N')^{1-\alpha} + (1 - \delta)]$$

LL (replacing $w(K)$ from N^D)

$$\frac{\chi}{1 - n} = \frac{(1 - \alpha) A K^{\alpha} N^{-\alpha}}{c}$$

2. Planner

EE

$$\frac{1}{C} = \beta \frac{1}{C'} [\alpha A(K')^{\alpha-1} (N')^{1-\alpha} + (1 - \delta)] - \beta [D'(P') F'(Y') \alpha A(K')^{\alpha-1} (N')^{1-\alpha}]$$

LL

$$\frac{\chi}{1 - N} = \frac{(1 - \alpha) A K^{\alpha} N^{-\alpha}}{C} - D'(F(Y)) F'(Y) (1 - \alpha) A K^{\alpha} N^{-\alpha}$$

3. EE

RCE

$$\frac{1}{c} = \beta \frac{1}{c'} [\alpha A(K')^{\alpha-1} (N')^{1-\alpha} + (1 - \delta)]$$

Planner

$$\frac{1}{C} = \beta \frac{1}{C'} [\alpha A(K')^{\alpha-1} (N')^{1-\alpha} + (1 - \delta)] - \underbrace{\beta D'(P') F'(Y') \alpha A(K')^{\alpha-1} (N')^{1-\alpha}}_{\text{disutility of extra pollution next period}}$$

- ▶ Note that they differ by the term in the right, which denotes the discounted disutility of having to experience higher pollution in the next period due to higher production as a function of higher savings.
- ▶ The planner internalizes the disutility of extra production that happens through pollution, while households ignore it in the competitive equilibrium.
- ▶ In the competitive equilibrium households oversave and overconsume compared to the social optimum.

3. LL

RCE

$$\frac{\chi}{1-n} = \frac{(1-\alpha)AK^{\alpha}N^{-\alpha}}{c}$$

Planner

$$\frac{\chi}{1-N} = \frac{(1-\alpha)AK^{\alpha}N^{-\alpha}}{C} - \underbrace{D'(P)F'(Y)(1-\alpha)AK^{\alpha}N^{-\alpha}}_{\text{disutility of extra pollution}}$$

- ▶ Note that they differ by the term in the right, which denotes the disutility of having to experience higher pollution today due to higher production as a function of higher hours worked.
- ▶ The planner internalizes the disutility of extra production that happens through pollution, while households ignore it in the competitive equilibrium.
- ▶ In the competitive equilibrium households overwork compared to the social optimum.

Appendix

When Blackwell's Conditions do not work

Steady State Proof

Practice Question: Midterm 2022

Pie Eating Problem. Consider the sequence problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\pi_{t+1} = \pi_t - c_t$$

$$0 < c_t \leq \pi_t$$

where $0 < \beta < 1$ and $\pi_0 > 0$ is given. Preferences are in the CRRA family,

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}}{1-\gamma} & \text{if } 0 < \gamma < \infty, \gamma \neq 1 \\ \ln(c_t) & \text{if } \gamma = 1 \end{cases}$$

1. Express the problem recursively using the FE representation.
2. Define the objects which constitute a solution to the FE problem.
3. Explain why you cannot directly apply Blackwell's Sufficient Conditions to prove that there exists a unique solution.

Hint: Identify an assumption that is violated with CRRA utility.

Solution: Pie Eating Problem I

1. Express the problem recursively using the FE representation.

The SP is

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\pi_t)$$

$$s.t. \quad \pi_{t+1} = \pi_t - c_t$$

$$0 < c_t \leq \pi_t$$

The FE is

$$v(\pi) = \max_{\{\pi'\}} \{u(c) + \beta v(\pi')\}$$

$$s.t. \quad \pi' = \pi - c$$

$$0 < c \leq \pi$$

Which we can also write:

$$v(\pi) = \max_{\{\pi'\}} \{u(\pi - \pi') + \beta v(\pi')\}$$

Solution: Pie Eating Problem II

2. Define the objects which constitute a solution to the FE problem.

The solution to this FE problem is a value function $v(\pi)$ which denotes the discounted maximized lifetime utility for an individual with wealth level π at the present period; and policy functions $c(\pi)$ and $\pi'(\pi)$, which denote the optimal choices of consumption and future assets for an individual with wealth level π at the present period.

Solution: Pie Eating Problem III

3. Explain why you cannot directly apply Blackwell's to prove that there exists a unique solution.
 - ▶ You cannot directly apply Blackwell's Sufficient Conditions because $u(c) : \mathbb{R} \rightarrow \mathbb{R}$ is not bounded (as c approaches 0, $u(c) \rightarrow -\infty$ and when $\gamma \rightarrow \infty$, $u(c) \rightarrow \infty$ if $c > 1$).

Steady state k^{ss}

Exercise: Consider the Neoclassical Model

$$v(k_t) = \max_{k_{t+1}} \{u(f(k_t) + (1 - \delta)k_t - k_{t+1}) + \beta v(k_{t+1})\}$$

1. Using the FONC on k_{t+1} and the Envelope Condition, find the Euler Equation. (We did it in the discussion!)
2. Assume a steady state capital (k^{ss}) exists. Find an expression for it. (Try this on your own from EE)

Now the real proof begins:

3. Using the concavity of v and f , prove that the expression you found in 2. is always a steady state.
4. Show that k_t converges monotonically to k^{ss}

Solution: Steady state k^{ss} I

3. Using the concavity of v and f , prove that the expression you found in 2. is always a steady state.

Claim: If $k^{ss} = f'^{-1}(\beta^{-1} - 1 + \delta)$, then k^{ss} is a steady state ($g(k^{ss}) = k^{ss}$)

Proof: Concavity of v implies:

- ▶ If $g(k) \geq k \implies v'(g(k)) \leq v'(k)$
- ▶ If $g(k) \leq k \implies v'(g(k)) \geq v'(k)$
- ▶ Under either:

$$[v'(g(k)) - v'(k)] [g(k) - k] \leq 0$$

By substituting

- ▶ $v'(g(k))$ from the FONC $\{k_{t+1}\}$: $u'(c) = \beta v'(g(k)) \implies v'(g(k)) = \beta^{-1} u'(c)$
- ▶ $v'(k)$ from the Envelope Condition: $v'(k) = u'(c)(f'(k) + 1 - \delta)$

We get

- ▶ $[u'(c)\beta^{-1} - u'(c)(f'(k) + 1 - \delta)] [g(k) - k] \leq 0$
- ▶ Which simplifies to $[\beta^{-1} - (f'(k) + 1 - \delta)] (g(k) - k) \leq 0$
- ▶ Multiplying by -1 : $[f'(k) + (1 - \delta) - \beta^{-1}] (g(k) - k) \geq 0$

Solution: Steady state k^{ss} II

We arrived to:

$$\underbrace{[f'(k) - \beta^{-1} + (1 - \delta)]}_A \underbrace{[g(k) - k]}_B \geq 0 \quad (1)$$

By strict concavity of f :

- ▶ If $k > k^{ss}$: $f'(k) < \beta^{-1} - (1 - \delta) = f'(k^{ss})$ ($A < 0$)
- ▶ If $k < k^{ss}$: $f'(k) > \beta^{-1} - (1 - \delta) = f'(k^{ss})$ ($A > 0$)

Why? Recall k^{ss} is defined by $f'(k^{ss}) = \beta^{-1} - (1 - \delta)$

Note that:

- ▶ If $g(k) > k$ ($B > 0$): For $A \times B \geq 0$ we need $A \geq 0$, but if $k > k^{ss}$, $A < 0$ a contradiction with (1)
- ▶ If $g(k) < k$ ($B < 0$): For $A \times B \geq 0$ we need $A \leq 0$, but if $k < k^{ss}$ $A > 0$ a contradiction with (1)
- ▶ We showed $g(k) = k = k^{ss}$, which concludes the proof □

Solution: Steady state k^{ss} III

4. Show that k_t converges monotonically to k^{ss}

Claim: $g(k_t)$ converges monotonically to k^{ss} .

$$k > k^{ss} \implies k^{ss} < g(k) < k$$

$$k < k^{ss} \implies k < g(k) < k^{ss}$$

Proof: Recall equation (1)

$$\underbrace{[f'(k) - \beta^{-1} + (1 - \delta)]}_A \underbrace{[g(k) - k]}_B \geq 0$$

Recall at k^{ss} , $f'(k^{ss}) = \beta^{-1} - (1 - \delta)$

By strict concavity of f :

- ▶ If $k > k^{ss}$: $f'(k) < \beta^{-1} - (1 - \delta)$ ($A < 0$) so it must be that $g(k) - k < 0$ ($B < 0$) "capital stock shrinks"
- ▶ If $k < k^{ss}$: $f'(k) > \beta^{-1} - (1 - \delta)$ ($A > 0$) so it must be that $g(k) - k > 0$ ($B > 0$) "capital stock grows"

□