Econ 210A
Discussion Section N°2

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Plan for today

SP vs FE

Bellman Equation vs Bellman Operator

Contraction Mapping Theorem & Blackwell's Sufficient Conditions

Practice Question 1: Dynamic Programming and Optimal Stopping (Midterm 2024)

Practice Question 2: Dynamic Programming in the Neoclassical Model (Midterm 2023)

Sequential Problem (SP)

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

st

$$x_{t+1} \in \Gamma(x_t)$$

$$x_0 \text{ given}$$

where

- $x \in X \subset \mathbb{R}^m$ is the state vector in the state space
- $F : X \times X \to \mathbb{R}$ is the instantaneous return function
- $\Gamma : X \to X$ is the correspondence for feasible values of the state

Solution: $\{x_{t+1}\}_{t=0}^{\infty}$

Functional Equation (FE)

$$V(x_t) = \sup_{x_{t+1} \in \Gamma(x_t)} \left\{ F(x_t, x_{t+1}) + \beta V(x_{t+1}) \right\}$$

where

- $F(x_t, x_{t+1})$: flow payoff
- $V(x_t)$: current value function
- $V(x_{t+1})$: continuation value

Solution: $V(x)$ and $G(x)$

$= y \in \Gamma(x) : V(x) = \sup F(x, y) + \beta V(y)$

Example: Neoclassical BSP Problem

Sequential Problem

$$V(K_0) = \max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(K_t) - K_{t+1})$$

st

$$K_{t+1} \in \Gamma(K_t) = [0, f(K_t)]$$
$$K_0 \text{ given}$$

Solution: $\{K_{t+1}\}_{t=0}^{\infty}$

Functional Equation

$$V(K_t) = \max_{K_{t+1} \in \Gamma(K_t)} u(f(K_t) - K_{t+1}) + \beta V(K_{t+1})$$

Solution: $V(K)$ and $G(K) = K'$

Solution to FE: Key Definitions

1. **Functional / Bellman Equation (FE)**

$$v(x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}$$

2. **Functional / Bellman Operator** T For any function $h(x)$,

$$T[h](x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta h(x') \right\}$$

3. **Fixed Point of** T A function w such that

$$T[w](x) = w(x) \quad \forall x.$$

Key Idea: The value function v solves the FE $\iff$ v is a fixed point of the operator T.
So, proving existence of v reduces to proving T has a fixed point.

Solution to FE: Contraction Mapping Theorem

Contraction Mapping Theorem If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then:

1. T has exactly one fixed point $V \in S$, and
2. For any $v_0 \in S, \quad \rho(T^n v_0, V) \leq \beta^n \rho(v_0, V) \quad \forall n = 0, 1, 2, \ldots$

Implications for Dynamic Programming:

- Existence and uniqueness of a solution v to our dynamic problem.
- A constructive method to compute it: **Value Function Iteration**.

Next: Check the "IF" conditions!

a. (S, ρ) is a complete metric space
b. $T : S \to S$ is a contraction mapping with modulus β

"(S, ρ) is a Complete Metric Space"

General concepts:

- **Metric space:** A pair (S, ρ) where ρ is a distance on S.
- **Cauchy sequence:** A sequence whose elements get *arbitrarily close to each other* as it progresses.

$$\{x_n\} \text{ is Cauchy if } \forall \epsilon > 0,\ \exists N \text{ s.t. } m, n > N \implies \rho(x_m, x_n) < \epsilon.$$

- **Complete metric space:** Every Cauchy sequence in S converges to an element of S.

Our case:

- S: space of bounded continuous functions on $X \subset \mathbb{R}^m$, often called $C_b(X)$
- ρ: supremum norm

$$\rho(f, g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$$

- This space (S, ρ) is complete.

Detour: Why Continuous Bounded Functions?

- Assumption (S&L 4.4): The one-period return function $F(x, y)$ is **continuous** and **bounded**.
- **Bounded:** f is bounded if $\exists M \geq 0$ such that

$$|f(x)| \leq M \quad \forall x \in X.$$

- If F is bounded by B, then the value function is also bounded:

$$|v^*(x)| \leq \frac{B}{1-\beta}.$$

- **Takeaway:** It is natural to restrict attention to the space of *bounded continuous functions*, where the Bellman operator acts and where we can prove existence/uniqueness.

"$T : S \to S$" or Why $T : S \to S$ is a Self-Map

Goal: Show that if $h(x) \in S$ (bounded, continuous), then $T[h](x) \in S$.

1. **Continuity**
   - By the *Maximum Theorem*: if
   $$j(x) = \max_{y \in \Gamma(x)} k(x, y),$$
   with k continuous and Γ compact-valued and continuous, then $j(x)$ is continuous.
   - Here: $k(x, y) = F(x, y) + \beta h(y)$ and $j(x) = T[h](x)$.
   - $\Rightarrow$ $T[h]$ is continuous.

2. **Boundedness**
   - Assumption S&L 4.4: F is bounded.
   - By construction h is bounded.
   - $\Rightarrow$ $T[h]$ is bounded.

Conclusion: $T[h]$ is continuous and bounded whenever h is. $\Rightarrow T : S \to S$ is a self-map.

"$T : S \to S$ is a contraction mapping with modulus β"

Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself.

Contraction mapping: T is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$ we have

$$\rho(Tx, Ty) \leq \beta\rho(x, y).$$

$$\forall x, y \in S$$

Blackwell's sufficient conditions

Let T be an operator defined over the space of bounded functions on X, $B(X)$. $T : B \to B$ is a contraction mapping with modulus β if it satisfies:

1. Monotonicity: For $f, g \in B$, $f(x) \leq g(x) \quad \forall x \in X \implies Tf(x) \leq Tg(x)$
2. Discounting: $\forall f \in B$, $c \geq 0$, $T(f + c)(x) \leq Tf(x) + \beta c$

Conclusion: From SP to Existence of Solution

- The **Sequential Problem (SP)** and the **Functional Equation (FE)** are equivalent formulations of dynamic optimization.
- The key object is the **Bellman Equation**, where the value function v is characterized as a **fixed point** of the Bellman operator T.
- By showing that T is a contraction mapping (via Blackwell's conditions) on the space of bounded continuous functions, we prove:
  - Existence of a solution v to the FE.
  - Uniqueness of that solution.
  - Convergence of iterative methods ($T^n v_0 \to v$).
- **Takeaway:** Dynamic programming ensures that well-behaved dynamic problems can be solved with simple recursive methods, and the solution is unique and stable.

Practice Question 1: Dynamic Programming and Optimal Stopping

An agent draws a random offer $x \sim \mathcal{U}[0,1]$. After observing x, the agent decides to either accept the offer and receive x (ending the game), or to reject the offer and draw another offer x' in the following period.
All offers are independent and the agent discounts future payoffs using discount factor β.
Offers continue in perpetuity until the agent accepts one.
Consequently, we can express the agent's problem recursively using the Bellman,

$$v(x) = \max \left\{ x \, , \, \beta \mathbb{E} \left[v(x') \right] \right\} \tag{1}$$

where expectations are over tomorrow's unknown offer, $x' \sim U(0,1)$.

A. Explain formally what it means to find an optimal policy which solves this problem.
What is the relationship between the policy function and $v(x)$?

Optimal policy $g(x)$: mapping from state variable (offer x) to action space $\{reject, accept\}$.

Value function $v(x)$: represents the value of having offer x under the optimal policy $g(x)$.

Practice Question 1: Dynamic Programming and Optimal Stopping

... Consequently, we can express the agent's problem recursively using the Bellman,

$$v(x) = \max \left\{ x \,,\, \beta \mathbb{E}\left[v(x') \right] \right\} \tag{2}$$

where expectations are over tomorrow's unknown offer, $x' \sim U(0,1)$.

B. Prove that there exists a unique solution to the above problem [*HINT: Recall Blackwell's Sufficient Conditions*]

Monotonicity: $f \geq g \implies T[f] \geq T[g]$

Here: $T[f](x) = \max\{x, \beta\mathbb{E}[f(x')]\} \geq \max\{x, \beta\mathbb{E}[g(x')]\} = T[g](x)$

Discounting: $T[f+a](x) \leq T[f](x) + \beta a$

Here: $T[f+a](x) = \max\{x, \beta\mathbb{E}[f(x')] + \beta a\} \leq \max\{x, \beta\mathbb{E}[f(x')]\} + \beta a = T[f](x) + \beta a$

$\implies$ By Blackwell's theorem:

- T is a contraction mapping with modulus β,
- v is the unique fix point of T and the solution to the FE.

Practice Question 2: Dynamic Programming in the Neoclassical Model

Consider an economy populated by infinitely lived households with forward-looking preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

where preferences are of the KPR type

$$u(c, n) = log(c) - \frac{1}{1+\kappa}(n)^{1+\kappa}$$

subject to an aggregate resource constraint

$$c_t + k_{t+1} \leq A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t$$

where $c_t, k_{t+1}, n_t \geq 0$, A is aggregate TFP, and k_0 is given.

A. Formulate the social planner's problem recursively.

B. Define formally what constitutes a solution to the problem.

C. Prove that there exists a unique solution to the social planner's problem.

Practice Question 2: Dynamic Programming in the Neoclassical Model

A. Formulate the social planner's problem recursively.

The SP is

$$\max_{\{k_{t+1}, n_t, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \frac{1}{1+\kappa}(n_t)^{1+\kappa} \right]$$

$$s.t. \quad c_t + k_{t+1} \leq Ak_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t$$

So the FE problem is:

$$\begin{aligned} v(k) \quad &= \quad \max_{c,n,k'} \quad \left\{ log(c) - \frac{1}{1+\kappa}(n)^{1+\kappa} + \beta v(k') \right\} \\ s.t. \quad & c + k' \leq Ak^{\alpha} n^{1-\alpha} + (1-\delta)k \end{aligned}$$

Practice Question 2: Dynamic Programming in the Neoclassical Model

B. Define formally what constitutes a solution to the problem.

A **solution to the Recursive Planner's Problem** is

- A value function $v(k)$,
- policy functions $c(k), n(k), k'(k)$,

that maximize discounted social lifetime utility such that the resource constraint is satisfied

$$c + k' \leq Ak^{\alpha} n^{1-\alpha} + (1-\delta)k$$

Practice Question 2: Dynamic Programming in the Neoclassical Model

C. Prove that there exists a unique solution to the social planner's problem.

Define the operator T:

$$T[v](k) = \max_{\{n,k'\}} \left\{ \log\left(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k'\right) - \frac{1}{1+\kappa}n^{1+\kappa} + \beta v(k') \right\}$$

To prove v is the unique solution to the problem, we will prove T is a contraction mapping with modulus β.

1. **Monotonicity**

   Assume $v(k') \geq w(k') \quad \forall k'$, then $\log(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k') - \frac{1}{1+\kappa}(n)^{1+\kappa} + \beta v(k') \geq \log(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k') - \frac{1}{1+\kappa}(n)^{1+\kappa} + \beta w(k') \quad \forall k'$.

   Which will also hold for n and k' that maximize each expression:
   $\max_{\{n,k'\}} \log(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k') - \frac{1}{1+\kappa}(n)^{1+\kappa} + \beta v(k') \geq$
   $\max_{\{n,k'\}} \log(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k') - \frac{1}{1+\kappa}(n)^{1+\kappa} + \beta w(k')$

   Which is equivalent to: $T[v](k) \geq T[w](k)$.

   $\implies$ T satisfies Monotonicity.

Practice Question 2: Dynamic Programming in the Neoclassical Model

C. Prove that there exists a unique solution to the social planner's problem.

2. **Discounting**

   For scalar a,

$$T[v+a](k) = \max_{\{n,k'\}} \left\{ \log\left(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k'\right) - \frac{1}{1+\kappa}n^{1+\kappa} + \beta[v(k') + a)] \right\}$$

$$= \max_{\{n,k'\}} \left\{ \log\left(Ak^{\alpha}n^{1-\alpha} + (1-\delta)k - k'\right) - \frac{1}{1+\kappa}n^{1+\kappa} + \beta v(k') \right\} + \beta a$$

$$= T[v](k) + \beta a$$

   $\implies$ T satisfies discounting.

We proved, by Blackwell's theorem, that T is a contraction mapping with modulus β. Therefore, v is the unique fix point of T and the solution to the FE.

General Assumptions	**BSP Neoclassical Model**
A3 X convex	$X = [0, \bar{k}]$
$\Gamma : X \to X$ non empty, compact, cts	$\Gamma(k_t) = [0, f(k_t) + (1-\delta)k_t]$
A4 $F : X \times X \to \mathbb{R}$ cts, bounded	$F(x_t, x_{t+1}) = u(f(k_t) + (1-\delta)k_t - k_{t+1})$
Result 1 Bellman operator is contraction mapping $\implies$ v (solution to FE) $\exists$, unique	
A5 F strictly increasing in first l arguments	u strictly increasing in k_t
A6 Γ monotone ($x \leq \bar{x} \implies \Gamma(x) \subseteq \Gamma(\bar{x})$)	$\Gamma(k_t)$ monotone in k_t
Result 2 v is strictly increasing	
A7 F strictly concave	u strictly concave
A8 Γ convex	$\Gamma(k_t)$ convex
Result 3 v is strictly concave, g (policy correspondence) is cts, single valued	
A9 F cts differentiable on interior of X	u cts differentiable
Result 4 Envelope theorem:	
$v(x) = F(x, g(x)) + \beta v(g(x))$ $v'(x) = F_1(x, g(x))$	$v(k) = u(f(k) + (1-\delta)k - g(k)) + \beta v(g(k))$ $v'(k) = u'(f(k) + (1-\delta)k - g(k))$
Result 5 (not needing A6,A8) Sufficiency of EE and TVC for solving SP	