Econ 210A
Discussion Section N°1

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Plan for today

Introduction to 210A and the Macro sequence

The BSP Sequential Problem

The BSP Recursive Problem

Overview of the Macro sequence

210 A

- **Methods**: Constrained optimization, Dynamic programming
- **Language**: Characterization of macro models, Neoclassical assumptions, BSP, Types of decentralized market equilibria
- **Topics**: Growth, RBC, Ramsey Policies, Aiyagari

210 B

- **Methods**: Optimal control, Dynamic programming in cts time
- **Language**: Heterogeneity, Frictions, OLG
- **Topics**: Asset pricing, Incomplete markets (again Aiyagari), Search and Matching

210 C

- **All topics**: Financial frictions, Credit supply, Investment, Money, New Keynesian Framework

210A and the Macro sequence

More on 210 A

- 4 (intense) problem sets
  - Goal is for you to learn
  - Useful to work on teams
  - Start now!
- Midterm and Final exam
  - Think of them as checkpoints towards the Qual

General advice

- Don't stress over grades, do worry about learning!
- Remember the PhD is a long run, not a sprint.
- Take care of yourself and your team.

Administrative

Regular schedule (starting Oct 16):

- Section: Thursdays, 3:00–4:20 PM (Atkinson 4004)
- Office Hours: Thursdays, 4:30–5:30 PM (Atkinson 4004)

First two weeks (Oct 2 & Oct 9):

- Section: Thursdays, 5:00–6:20 PM (SSB 107)
- Office Hours: Thursdays, 6:30–7:30 PM (SSB 107)

1st Problem Set: due **Saturday, Oct 11** via Gradescope

The Benevolent Social Planner (BSP) Problem

The Benevolent Social Planner (BSP) Problem

- **Not the government** — it is a modeling device.
- Modern macro builds from agents' optimal decisions (households, firms, government*) and their interaction in markets.
- Recall the **First Welfare Theorem**: a competitive equilibrium (CE) allocation is Pareto optimal (PO).
- So: in worlds where the theorem holds, we can find the market allocation by solving for the PO allocation.
- The BSP formulation provides a convenient way to characterize PO allocations.

But will this trick always work? 😐

Stay tuned for surprises!

Neoclassical Model BSP as a SP

↗ Benevolent social Planner

↳ sequential Problem

Solution: sequence $\{C_t, K_{t+1}\}_{t=0}^{T}$ that solve the following problem

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(C_t)$$

subject to

$$C_t + K_{t+1} \leq F(K_t) + (1-\delta)K_t$$
$$C_t,\ K_{t+1} > 0$$
$$K_0 \quad \text{given}$$

Neoclassical assumptions

Preferences: $\beta \in (0,1)$, $u(C_t)$ is twice cts differentiable and satisfies

$$u'(C_t) > 0 \quad u''(C_t) < 0 \quad \lim_{C_t \to 0} u'(C_t) = \infty$$

Technology: $\delta \in (0,1)$, $F(K_t)$ is twice cts differentiable and satisfies

$$F(0) = 0 \quad F'(K_t) > 0 \quad F''(K_t) < 0 \quad \lim_{K_t \to 0} F'(K_t) = \infty \quad \lim_{K_t \to \infty} F'(K_t) = 0$$

Neoclassical Model BSP as a SP: Solution

- The solution to the previous sequential problem (SP) is a sequence of $\{C_t, K_{t+1}\}_{t=0}^{T}$ that maximizes the discounted sum of lifetime utility, subject to resources constraints.
- Lets use constrained optimization to characterize this sequence...

(Review) Constrained Optimization: Lagrangian + KKT

$$\max_{x \in S} \; g(x)$$

subject to

$$h_i(x) \geq 0 \quad (i = 1, ..., I)$$

Lagrangian:

$$\mathcal{L}(x, \lambda) = g(x) + \sum_{i=1}^{I} \lambda_i h_i(x)$$

Result: Any $x^* \in S$ -an interior point solution to the problem- satisfies the KKT conditions.

KKT conditions:

1. FONC: $\nabla_x L(x, \lambda) = 0$
2. $h_i(x) \geq 0$ and $\lambda_i \geq 0$
3. Complementary Slackness: $\lambda_i h_i(x) = 0 \quad \forall i$
   - $\lambda_i > 0 \implies \mathrm{h}_i = 0$
   - $h_i(x) > 0 \implies \lambda_i = 0$
   - $h_i(x) = 0 \implies \lambda_i \geq 0$

***Assumptions**: g, h_i **are real-valued concave and cts functions and** S **is a convex set**.

Neoclassical Model BSP as a SP: Solution

Lagrangian

★ IMPORTANT : if a constraint must be satisfied in every period, then its not one constraint but T+1 or ∞ (depending on how many periods this problem lasts!

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u(C_t) + \lambda_t \left[F(K_t) + (1-\delta)K_t - C_t - K_{t+1} \right] + \mu_t K_{t+1} \right\}$$

Therefore we need to include all of them in the Lagrangian

FONC

★ TIP : include $\lambda_t h_t$ inside the time sumation and you'll be fine

$$\frac{\partial \mathcal{L}}{\partial C_t} : \quad u'(C_t) - \lambda_t = 0 \qquad \forall t = 0, \dots, T$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} : \quad -\lambda_t + \beta \lambda_{t+1} \left[F'(K_{t+1}) + (1-\delta) \right] + \mu_t = 0 \qquad t = 0, \dots, T-1$$

$$: \quad -\lambda_T + \mu_T = 0 \qquad t = T$$

★ Bcs "$\beta \lambda_{T+1} [F'(K_{T+1}) + (1-\delta)]$" does not exist!!

Complementary slackness

$$\lambda_t : \quad \lambda_t \left[F(K_t) + (1-\delta)K_t - C_t - K_{t+1} \right] = 0 \qquad \forall t = 0, \dots, T$$

$$\mu_t : \quad \mu_t K_{t+1} = 0 \qquad \forall t = 0, \dots, T$$

Note that in the previous derivation we had

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u(c_t) + \lambda_t \left[F(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right] + \mu_t k_{t+1} \right\}$$

I told you its important to have $\lambda_t(\cdot)$ and $\mu_t(\cdot)$ inside the time sumation, but it is up to you if you want it inside or outside $\{\ \}$

This is perfectly fine too:

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ \beta^t u(c_t) + \lambda_t \left[F(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right] + \mu_t k_{t+1} \right\}$$

you will get the same result, only the interpretation of λ_t and μ_t change. In this formulation they embed the time discouting, in the previous they are net of time discounting, which is done by β^t

Neoclassical Model BSP as a SP: Solution

Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u(C_t) + \lambda_t \left[F(K_t) + (1-\delta)K_t - C_t - K_{t+1} \right] + \mu_t K_{t+1} \right\}$$

FONC

$$\frac{\partial \mathcal{L}}{\partial C_t} : \quad u'(C_t) - \lambda_t = 0 \qquad \forall t = 0, \ldots, T$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} : \quad -\lambda_t + \beta \lambda_{t+1} \left[F'(K_{t+1}) + (1-\delta) \right] + \mu_t = 0 \qquad t = 0, \ldots, T-1$$

$$: \quad -\lambda_T + \mu_T = 0 \qquad t = T$$

Complementary slackness

$$\lambda_t : \quad \lambda_t \left[F(K_t) + (1-\delta)K_t - C_t - K_{t+1} \right] = 0 \qquad \forall t = 0, \ldots, T$$

$$\mu_t : \quad \mu_t K_{t+1} = 0 \qquad \forall t = 0, \ldots, T$$

What can we learn??

We know $K_{t+1} > 0$ for $t = 0, \ldots, T-1$ because o.w. $C_{t+1} = 0$ a contradiction with optimality...

$$\implies \mu_t = 0 \quad t = 0, \ldots, T-1, \quad \mu_T \geq 0$$

$\Downarrow$ it must be the case that $\mu_t = 0$ for $t = 0, \ldots, T-1$!!

Neoclassical BSP Problem: Solution

We want $\{C_t^*, K_{t+1}^*\}_{t=0}^{T}$.

(1) Euler Equation (EE) from first two FONC:

$$u'\big(F(K_t)+(1-\delta)K_t-K_{t+1}\big) = \beta\, u'\big(F(K_{t+1})+(1-\delta)K_{t+1}-K_{t+2}\big)\cdot\big(F'(K_{t+1})+1-\delta\big),$$
$$\forall t = 0,\ldots,T-1$$

(2) Terminal Condition from "3rd" FONC + slackness:

$$\mu_T = \lambda_T = u'(C_T) > 0 \;\;\Rightarrow\;\; \mu_T K_{T+1} = 0 \quad\Longrightarrow\quad K_{T+1} = 0$$

- Finite horizon: solve recursively for $K_1,\ldots,K_T$ by backward induction.
- Infinite horizon: replace with a **Transversality Condition (TVC)**.

Takeaway: EE links K_t, K_{t+1}, K_{t+2}; terminal/TVC pins down the path.

BSP Formulation of Neoclassical Model

Some natural questions remain:

- Existence: does a solution even exist?
- Uniqueness: is there only one?
- Method: how can we prove it? $\Rightarrow$ Dynamic Programming!

Detour : Necessary vs Sufficient conditions

In a simple optimization problem we call the first order derivative $= 0$ a necessary condition for an optimum, and for sufficiency we check second order conditions.

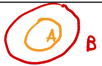
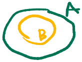
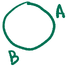
In our SP, we just found that EE + TVC are **necessary** for an optimum

WHAT DOES THIS MEAN ? = CASE 1 BELOW

Define A : " $\{c_t, k_{t+1}\}_{t=0}^{T/\infty}$ is the solution to the SP "

B : " $\{c_t, k_{t+1}\}_{t=0}^{T/\infty}$ satisfies EE and TC/TVC "

In english: "EE+TVC are necessary for optimality"

	LOGIC	SETS
1 B is necessary for A	$A \Rightarrow B$	$A \subseteq B$ 
2 B is sufficient for A	$A \Leftarrow B$	$A \supseteq B$ 
3 B is necessary and sufficient for A	$A \Leftrightarrow B$	$A = B$ 

The BSP Recursive Problem

- The BSP can be written as a **Sequential Problem (SP)**: choose $\{C_t, K_{t+1}\}_{t=0}^{\infty \text{ or } T}$ to maximize lifetime utility subject to constraints.
- Alternatively, use Dynamic Programming: the **Functional Equation (FE)** approach.
- Under mild assumptions:
  - SP $\Rightarrow$ solves FE (S&L Thm 4.2)
  - FE $\Rightarrow$ solves SP (S&L Thm 4.3)

The BSP Recursive Problem: Value Function

- Suppose we solved the SP for all K_0.
- Define the **value function**:

$$v(K_0) = \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \qquad \text{s.t. } C_t + K_{t+1} \leq F(K_t),\ K_0 \text{ given.}$$

- Separate out the first term in the objective function:

$$\begin{aligned} v(K_0) &= \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \left[u(C_0) + \beta \sum_{t=0}^{\infty} \beta^t u(C_{t+1}) \right] \\ &= \max_{C_0, K_1} \left\{ u(C_0) + \beta \max_{\{C_{t+1}, K_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_{t+1}) \right\} \end{aligned}$$

- Recognize the continuation as the value from K_1:

$$v(K_0) = \max_{C_0, K_1} \left\{ u(C_0) + \beta v(K_1) \right\} \qquad \text{s.t. } C_0 + K_1 \leq F(K_0), K_0 \text{ given.}$$

This is the Bellman Equation.

The BSP Recursive Problem

$$\boxed{v(K) = \max_{C,K'} \left\{ u(C) + \beta v(K') \right\} \quad \text{s.t. } C + K' \leq F(K)}$$

- The solution is not a specific C or K' for one K, but a **policy function** $g(K)$ mapping each state to optimal choices.
- The associated **value function** $v(K)$ gives the maximum attainable utility.

Neoclassical Model BSP as a FE

Solution: functions $v(K), g(K)$ that solve the following problem

$$v(K) \quad = \quad \max_{C,K'} \quad u(C) + \beta v(K')$$

subject to

$$C + K' \leq F(K) + (1 - \delta)K$$
$$C,\ K' > 0$$

Neoclassical assumptions

Preferences: $\beta \in (0,1)$, $u(C)$ is twice cts differentiable and satisfies

$$u'(C) > 0 \quad u''(C) < 0 \quad \lim_{C \to 0} u'(C) = \infty$$

Technology: $\delta \in (0,1)$, $F(K)$ is twice cts differentiable and satisfies

$$F(0) = 0 \quad F'(K) > 0 \quad F''(K) < 0 \quad \lim_{K \to 0} F'(K) = \infty \quad \lim_{K \to \infty} F'(K) = 0$$

Roadmap

- **Today:** We saw that Euler Equation (EE) + Transversality Condition (TVC) are *necessary* for a solution to the Sequential Problem (SP).
- **Next week:** We will show how the Functional Equation (FE) provides *necessary and sufficient* conditions.
- **Appendix:** Contains a preview of FE and supporting material — useful for your homework.

Appendix

Sequential Problem (SP)

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

st

$$x_{t+1} \in \Gamma(x_t)$$

$$x_0 \text{ given}$$

where

- $x \in X \subset \mathbb{R}^m$ is the state vector in the state space
- $F : X \times X \to \mathbb{R}$ is the instantaneous return function
- $\Gamma : X \to X$ is the correspondence for feasible values of the state

Solution: $\{x_{t+1}\}_{t=0}^{\infty}$

Functional Equation (FE)

$$V(x_t) = \sup_{x_{t+1} \in \Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta V(x_{t+1})\}$$

where

- $F(x_t, x_{t+1})$: flow payoff
- $V(x_t)$: current value function
- $V(x_{t+1})$: continuation value

Solution: $V(x)$ and $G(x)$
$= y \in \Gamma(x) : V(x) = \sup F(x, y) + \beta V(y)$

Solution to FE: Key Definitions

1. **Functional / Bellman Equation (FE)**

$$v(x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}$$

2. **Functional / Bellman Operator** T For any function $h(x)$,

$$T[h](x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta h(x') \right\}$$

3. **Fixed Point of** T A function w such that

$$T[w](x) = w(x) \quad \forall x.$$

Key Idea: The value function v solves the FE $\iff$ v is a fixed point of the operator T.
So, proving existence of v reduces to proving T has a fixed point.

Solution to FE: Contraction Mapping Theorem

Contraction Mapping Theorem If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then:

1. T has exactly one fixed point $V \in S$, and
2. For any $v_0 \in S$, $\quad \rho(T^n v_0, V) \leq \beta^n \rho(v_0, V) \quad \forall n = 0, 1, 2, \ldots$

Implications for Dynamic Programming:

- Existence and uniqueness of a solution v to our dynamic problem.
- A constructive method to compute it: **Value Function Iteration**.

Next: Check the "IF" conditions!

a. (S, ρ) is a complete metric space

b. $T : S \to S$ is a contraction mapping with modulus β

"(S, ρ) is a Complete Metric Space"

General concepts:

- **Metric space:** A pair (S, ρ) where ρ is a distance on S.
- **Cauchy sequence:** A sequence whose elements get *arbitrarily close to each other* as it progresses.

$$\{x_n\} \text{ is Cauchy if } \forall \epsilon > 0,\ \exists N \text{ s.t. } m, n > N \implies \rho(x_m, x_n) < \epsilon.$$

- **Complete metric space:** Every Cauchy sequence in S converges to an element of S.

Our case:

- S: space of bounded continuous functions on $X \subset \mathbb{R}^m$, often called $C_b(X)$
- ρ: supremum norm

$$\rho(f, g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$$

- This space (S, ρ) is complete.

Detour: Why Continuous Bounded Functions?

- Assumption (S&L 4.4): The one-period return function $F(x, y)$ is **continuous** and **bounded**.
- **Bounded:** f is bounded if $\exists M \geq 0$ such that

$$|f(x)| \leq M \quad \forall x \in X.$$

- If F is bounded by B, then the value function is also bounded:

$$|v^*(x)| \leq \frac{B}{1-\beta}.$$

- **Takeaway:** It is natural to restrict attention to the space of *bounded continuous functions*, where the Bellman operator acts and where we can prove existence/uniqueness.

"$T : S \to S$" or Why $T : S \to S$ is a Self-Map

Goal: Show that if $h(x) \in S$ (bounded, continuous), then $T[h](x) \in S$.

1. **Continuity**
   - By the *Maximum Theorem*: if
   $$j(x) = \max_{y \in \Gamma(x)} k(x, y),$$
   with k continuous and Γ compact-valued and continuous, then $j(x)$ is continuous.
   - Here: $k(x, y) = F(x, y) + \beta h(y)$ and $j(x) = T[h](x)$.
   - $\Rightarrow T[h]$ is continuous.
2. **Boundedness**
   - Assumption S&L 4.4: F is bounded.
   - By construction h is bounded.
   - $\Rightarrow T[h]$ is bounded.

Conclusion: $T[h]$ is continuous and bounded whenever h is.
$\Rightarrow T : S \to S$ is a self-map.

"$T : S \to S$ is a contraction mapping with modulus β"

Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself.

Contraction mapping: T is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$ we have

$$\rho(Tx, Ty) \leq \beta\rho(x, y).$$

$$\forall x, y \in S$$

Blackwell's sufficient conditions

Let T be an operator defined over the space of bounded functions on X, $B(X)$. $T : B \to B$ is a contraction mapping with modulus β if it satisfies:

1. Monotonicity: For $f, g \in B$, $f(x) \leq g(x) \quad \forall x \in X \implies Tf(x) \leq Tg(x)$
2. Discounting: $\forall f \in B$, $c \geq 0$, $T(f + c)(x) \leq Tf(x) + \beta c$

Conclusion: From SP to Existence of Solution

- The **Sequential Problem (SP)** and the **Functional Equation (FE)** are equivalent formulations of dynamic optimization.
- The key object is the **Bellman Equation**, where the value function v is characterized as a **fixed point** of the Bellman operator T.
- By showing that T is a contraction mapping (via Blackwell's conditions) on the space of bounded continuous functions, we prove:
  - Existence of a solution v to the FE.
  - Uniqueness of that solution.
  - Convergence of iterative methods ($T^n v_0 \to v$).
- **Takeaway:** Dynamic programming ensures that well-behaved dynamic problems can be solved with simple recursive methods, and the solution is unique and stable.