

Econ 210A

Discussion Section N°1

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Fall 2025

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Plan for today

Introduction to 210A and the Macro sequence

The BSP Sequential Problem

The BSP Recursive Problem

Overview of the Macro sequence

210 A

- ▶ **Methods:** Constrained optimization, Dynamic programming
- ▶ **Language:** Characterization of macro models, Neoclassical assumptions, BSP, Types of decentralized market equilibria
- ▶ **Topics:** Growth, RBC, Ramsey Policies, Aiyagari

210 B

- ▶ **Methods:** Optimal control, Dynamic programming in continuous time
- ▶ **Language:** Heterogeneity, Frictions, OLG
- ▶ **Topics:** Asset pricing, Incomplete markets (again Aiyagari), Search and Matching

210 C

- ▶ **All topics:** Financial frictions, Credit supply, Investment, Money, New Keynesian Framework

210A and the Macro sequence

More on 210 A

- ▶ 4 (intense) problem sets
 - ▶ Goal is for you to learn
 - ▶ Useful to work on teams
 - ▶ Start now!
- ▶ Midterm and Final exam
 - ▶ Think of them as checkpoints towards the Qual

General advice

- ▶ Don't stress over grades, do worry about learning!
- ▶ Remember the PhD is a long run, not a sprint.
- ▶ Take care of yourself and your team.

Administrative

Regular schedule (starting Oct 16):

- ▶ Section: Thursdays, 3:00–4:20 PM (Atkinson 4004)
- ▶ Office Hours: Thursdays, 4:30–5:30 PM (Atkinson 4004)

First two weeks (Oct 2 & Oct 9):

- ▶ Section: Thursdays, 5:00–6:20 PM (SSB 107)
- ▶ Office Hours: Thursdays, 6:30–7:30 PM (SSB 107)

1st Problem Set: due **Saturday, Oct 11** via Gradescope

The Benevolent Social Planner (BSP) Problem



Thanks to Carlos Goes for the picture!

The Benevolent Social Planner (BSP) Problem

- ▶ **Not the government** — it is a modeling device.
- ▶ Modern macro builds from agents' optimal decisions (households, firms, government*) and their interaction in markets.
- ▶ Recall the **First Welfare Theorem**: a competitive equilibrium (CE) allocation is Pareto optimal (PO).
- ▶ So: in worlds where the theorem holds, we can find the market allocation by solving for the PO allocation.
- ▶ The BSP formulation provides a convenient way to characterize PO allocations.

But will this trick always work? 😐

Stay tuned for surprises!

Neoclassical Model BSP as a SP

Solution: sequence $\{C_t, K_{t+1}\}_{t=0}^T$ that solve the following problem

$$\max_{\{C_t, K_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(C_t)$$

subject to

$$C_t + K_{t+1} \leq F(K_t) + (1 - \delta)K_t$$

$$C_t, K_{t+1} > 0$$

$$K_0 \text{ given}$$

Neoclassical assumptions

Preferences: $\beta \in (0, 1)$, $u(C_t)$ is twice cts differentiable and satisfies

$$u'(C_t) > 0 \quad u''(C_t) < 0 \quad \lim_{C_t \rightarrow 0} u'(C_t) = \infty$$

Technology: $\delta \in (0, 1)$, $F(K_t)$ is twice cts differentiable and satisfies

$$F(0) = 0 \quad F'(K_t) > 0 \quad F''(K_t) < 0 \quad \lim_{K_t \rightarrow 0} F'(K_t) = \infty \quad \lim_{K_t \rightarrow \infty} F'(K_t) = 0$$

Neoclassical Model BSP as a SP: Solution

- ▶ The solution to the previous sequential problem (SP) is a sequence of $\{C_t, K_{t+1}\}_{t=0}^T$ that maximizes the discounted sum of lifetime utility, subject to resources constraints.
- ▶ Lets use **constrained optimization** to characterize this sequence...

(Review) Constrained Optimization: Lagrangian + KKT

$$\max_{x \in S} g(x)$$

subject to

$$h_i(x) \geq 0 \quad (i = 1, \dots, l)$$

Lagrangian:

$$\mathcal{L}(x, \lambda) = g(x) + \sum_{i=1}^l \lambda_i h_i(x)$$

Result: Any $x^* \in S$ -an interior point solution to the problem- satisfies the KKT conditions.

KKT conditions:

1. FONC: $\nabla_x \mathcal{L}(x, \lambda) = 0$
2. $h_i(x) \geq 0$ and $\lambda_i \geq 0$
3. Complementary Slackness: $\lambda_i h_i(x) = 0 \quad \forall i$
 - ▶ $\lambda_i > 0 \implies h_i = 0$
 - ▶ $h_i(x) > 0 \implies \lambda_i = 0$
 - ▶ $h_i(x) = 0 \implies \lambda_i \geq 0$

***Assumptions:** g, h_i are real-valued concave and cts functions and S is a convex set.

Neoclassical Model BSP as a SP: Solution

Lagrangian

$$\mathcal{L} = \sum_{t=0}^T \beta^t \{u(C_t) + \lambda_t [F(K_t) + (1 - \delta)K_t - C_t - K_{t+1}] + \mu_t K_{t+1}\}$$

FONC

$$\frac{\partial \mathcal{L}}{\partial C_t} : \quad u'(C_t) - \lambda_t = 0 \quad \forall t = 0, \dots, T$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{t+1}} : \quad & -\lambda_t + \beta \lambda_{t+1} [F'(K_{t+1}) + (1 - \delta)] + \mu_t = 0 & t = 0, \dots, T - 1 \\ & -\lambda_T + \mu_T = 0 & t = T \end{aligned}$$

Complementary slackness

$$\lambda_t : \quad \lambda_t [F(K_t) + (1 - \delta)K_t - C_t - K_{t+1}] = 0 \quad \forall t = 0, \dots, T$$

$$\mu_t : \quad \mu_t K_{t+1} = 0 \quad \forall t = 0, \dots, T$$

Neoclassical Model BSP as a SP: Solution

Lagrangian

$$\mathcal{L} = \sum_{t=0}^T \beta^t \{u(C_t) + \lambda_t [F(K_t) + (1 - \delta)K_t - C_t - K_{t+1}] + \mu_t K_{t+1}\}$$

FONC

$$\frac{\partial \mathcal{L}}{\partial C_t} : \quad u'(C_t) - \lambda_t = 0 \quad \forall t = 0, \dots, T$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{t+1}} : \quad & -\lambda_t + \beta \lambda_{t+1} [F'(K_{t+1}) + (1 - \delta)] + \mu_t = 0 & t = 0, \dots, T - 1 \\ & -\lambda_T + \mu_T = 0 & t = T \end{aligned}$$

Complementary slackness

$$\lambda_t : \quad \lambda_t [F(K_t) + (1 - \delta)K_t - C_t - K_{t+1}] = 0 \quad \forall t = 0, \dots, T$$

$$\begin{aligned} \mu_t : \quad & \mu_t K_{t+1} = 0 & \forall t = 0, \dots, T \\ \implies & \mu_t = 0 \quad t = 0, \dots, T - 1, \quad \mu_T \geq 0 \end{aligned}$$

Neoclassical BSP Problem: Solution

We want $\{C_t^*, K_{t+1}^*\}_{t=0}^T$.

(1) **Euler Equation (EE)** from first two FONC:

$$u'(F(K_t) + (1-\delta)K_t - K_{t+1}) = \beta u'(F(K_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}) \cdot (F'(K_{t+1}) + 1 - \delta),$$
$$\forall t = 0, \dots, T-1$$

(2) **Terminal Condition** from "3rd" FONC + slackness:

$$\mu_T = \lambda_T = u'(C_T) > 0 \Rightarrow \mu_T K_{T+1} = 0 \implies K_{T+1} = 0$$

- ▶ Finite horizon: solve recursively for $K_1, \dots, K_T$ by backward induction.
- ▶ Infinite horizon: replace with a **Transversality Condition (TVC)**.

Takeaway: EE links K_t, K_{t+1}, K_{t+2} ; terminal/TVC pins down the path.

BSP Formulation of Neoclassical Model

Some natural questions remain:

- ▶ **Existence:** does a solution even exist?
- ▶ **Uniqueness:** is there only one?
- ▶ **Method:** how can we prove it? $\Rightarrow$ Dynamic Programming!

The BSP Recursive Problem

- ▶ The BSP can be written as a **Sequential Problem (SP)**: choose $\{C_t, K_{t+1}\}_{t=0}^{\infty \text{ or } T}$ to maximize lifetime utility subject to constraints.
- ▶ Alternatively, use Dynamic Programming: the **Functional Equation (FE)** approach.
- ▶ Under mild assumptions:
 - ▶ SP $\Rightarrow$ solves FE (S&L Thm 4.2)
 - ▶ FE $\Rightarrow$ solves SP (S&L Thm 4.3)

The BSP Recursive Problem: Value Function

- ▶ Suppose we solved the SP for all K_0 .
- ▶ Define the **value function**:

$$v(K_0) = \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \quad \text{s.t. } C_t + K_{t+1} \leq F(K_t), K_0 \text{ given.}$$

- ▶ Separate out the first term in the objective function:

$$\begin{aligned} v(K_0) &= \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \left[u(C_0) + \beta \sum_{t=0}^{\infty} \beta^t u(C_{t+1}) \right] \\ &= \max_{C_0, K_1} \left\{ u(C_0) + \beta \max_{\{C_{t+1}, K_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_{t+1}) \right\} \end{aligned}$$

- ▶ Recognize the continuation as the value from K_1 :

$$v(K_0) = \max_{C_0, K_1} \left\{ u(C_0) + \beta v(K_1) \right\} \quad \text{s.t. } C_0 + K_1 \leq F(K_0), K_0 \text{ given.}$$

This is the Bellman Equation.

The BSP Recursive Problem

$$v(K) = \max_{C, K'} \left\{ u(C) + \beta v(K') \right\} \quad \text{s.t. } C + K' \leq F(K)$$

- ▶ The solution is not a specific C or K' for one K , but a **policy function** $g(K)$ mapping each state to optimal choices.
- ▶ The associated **value function** $v(K)$ gives the maximum attainable utility.

Neoclassical Model BSP as a FE

Solution: functions $v(K), g(K)$ that solve the following problem

$$v(K) = \max_{C, K'} u(C) + \beta v(K')$$

subject to

$$\begin{aligned} C + K' &\leq F(K) + (1 - \delta)K \\ C, K' &> 0 \end{aligned}$$

Neoclassical assumptions

Preferences: $\beta \in (0, 1)$, $u(C)$ is twice cts differentiable and satisfies

$$u'(C) > 0 \quad u''(C) < 0 \quad \lim_{C \rightarrow 0} u'(C) = \infty$$

Technology: $\delta \in (0, 1)$, $F(K)$ is twice cts differentiable and satisfies

$$F(0) = 0 \quad F'(K) > 0 \quad F''(K) < 0 \quad \lim_{K \rightarrow 0} F'(K) = \infty \quad \lim_{K \rightarrow \infty} F'(K) = 0$$

Roadmap

- ▶ **Today:** We saw that Euler Equation (EE) + Transversality Condition (TVC) are *necessary* for a solution to the Sequential Problem (SP).
- ▶ **Next week:** We will show how the Functional Equation (FE) provides *necessary and sufficient* conditions.
- ▶ **Appendix:** Contains a preview of FE and supporting material — useful for your homework.

Appendix

SP vs FE

Bellman Equation vs Bellman Operator

Contraction Mapping Theorem & Blackwell's Sufficient Conditions

Conclusion

Sequential Problem (SP)

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

st

$$x_{t+1} \in \Gamma(x_t)$$

x_0 given

where

- ▶ $x \in X \subset \mathbb{R}^m$ is the state vector in the state space
- ▶ $F : X \times X \rightarrow \mathbb{R}$ is the instantaneous return function
- ▶ $\Gamma : X \rightarrow X$ is the correspondence for feasible values of the state

Solution: $\{x_{t+1}\}_{t=0}^{\infty}$

Functional Equation (FE)

$$V(x_t) = \sup_{x_{t+1} \in \Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta V(x_{t+1})\}$$

where

- ▶ $F(x_t, x_{t+1})$: flow payoff
- ▶ $V(x_t)$: current value function
- ▶ $V(x_{t+1})$: continuation value

Solution: $V(x)$ and $G(x)$

$$= y \in \Gamma(x) : V(x) = \sup F(x, y) + \beta V(y)$$

Solution to FE: Key Definitions

1. Functional / Bellman Equation (FE)

$$v(x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}$$

2. Functional / Bellman Operator T For any function $h(x)$,

$$T[h](x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta h(x') \right\}$$

3. Fixed Point of T A function w such that

$$T[w](x) = w(x) \quad \forall x.$$

Key Idea: The value function v solves the FE $\iff v$ is a fixed point of the operator T .

So, proving existence of v reduces to proving T has a fixed point.

Solution to FE: Contraction Mapping Theorem

Contraction Mapping Theorem If (S, ρ) is a complete metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β , then:

1. T has exactly one fixed point $V \in S$, and
2. For any $v_0 \in S$, $\rho(T^n v_0, V) \leq \beta^n \rho(v_0, V) \quad \forall n = 0, 1, 2, \dots$

Implications for Dynamic Programming:

- ▶ Existence and uniqueness of a solution v to our dynamic problem.
- ▶ A constructive method to compute it: **Value Function Iteration**.

Next: Check the “IF” conditions!

- a. (S, ρ) is a complete metric space
- b. $T : S \rightarrow S$ is a contraction mapping with modulus β

" (S, ρ) is a Complete Metric Space"

General concepts:

- ▶ **Metric space:** A pair (S, ρ) where ρ is a distance on S .
- ▶ **Cauchy sequence:** A sequence whose elements get *arbitrarily close to each other* as it progresses.

$$\{x_n\} \text{ is Cauchy if } \forall \epsilon > 0, \exists N \text{ s.t. } m, n > N \implies \rho(x_m, x_n) < \epsilon.$$

- ▶ **Complete metric space:** Every Cauchy sequence in S converges to an element of S .

Our case:

- ▶ S : space of bounded continuous functions on $X \subset \mathbb{R}^m$, often called $C_b(X)$
- ▶ ρ : supremum norm

$$\rho(f, g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$$

- ▶ This space (S, ρ) is complete.

Detour: Why Continuous Bounded Functions?

- ▶ Assumption (S&L 4.4): The one-period return function $F(x, y)$ is **continuous** and **bounded**.
- ▶ **Bounded:** f is bounded if $\exists M \geq 0$ such that

$$|f(x)| \leq M \quad \forall x \in X.$$

- ▶ If F is bounded by B , then the value function is also bounded:

$$|v^*(x)| \leq \frac{B}{1 - \beta}.$$

- ▶ **Takeaway:** It is natural to restrict attention to the space of *bounded continuous functions*, where the Bellman operator acts and where we can prove existence/uniqueness.

" $T : S \rightarrow S$ " or Why $T : S \rightarrow S$ is a Self-Map

Goal: Show that if $h(x) \in S$ (bounded, continuous), then $T[h](x) \in S$.

1. Continuity

- ▶ By the *Maximum Theorem*: if

$$j(x) = \max_{y \in \Gamma(x)} k(x, y),$$

with k continuous and Γ compact-valued and continuous, then $j(x)$ is continuous.

- ▶ Here: $k(x, y) = F(x, y) + \beta h(y)$ and $j(x) = T[h](x)$.
- ▶ $\Rightarrow T[h]$ is continuous.

2. Boundedness

- ▶ Assumption S&L 4.4: F is bounded.
- ▶ By construction h is bounded.
- ▶ $\Rightarrow T[h]$ is bounded.

Conclusion: $T[h]$ is continuous and bounded whenever h is.
 $\Rightarrow T : S \rightarrow S$ is a self-map.

" $T : S \rightarrow S$ is a contraction mapping with modulus β "

Let (S, ρ) be a metric space and $T : S \rightarrow S$ be a function mapping S into itself.

Contraction mapping: T is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$ we have

$$\rho(Tx, Ty) \leq \beta \rho(x, y).$$

$$\forall x, y \in S$$

Blackwell's sufficient conditions

Let T be an operator defined over the space of bounded functions on X , $B(X)$.

$T : B \rightarrow B$ is a contraction mapping with modulus β if it satisfies:

1. Monotonicity: For $f, g \in B$, $f(x) \leq g(x) \quad \forall x \in X \implies Tf(x) \leq Tg(x)$
2. Discounting: $\forall f \in B, c \geq 0, T(f + c)(x) \leq Tf(x) + \beta c$

Conclusion: From SP to Existence of Solution

- ▶ The **Sequential Problem (SP)** and the **Functional Equation (FE)** are equivalent formulations of dynamic optimization.
- ▶ The key object is the **Bellman Equation**, where the value function v is characterized as a **fixed point** of the Bellman operator T .
- ▶ By showing that T is a contraction mapping (via Blackwell's conditions) on the space of bounded continuous functions, we prove:
 - ▶ Existence of a solution v to the FE.
 - ▶ Uniqueness of that solution.
 - ▶ Convergence of iterative methods ($T^n v_0 \rightarrow v$).
- ▶ **Takeaway:** Dynamic programming ensures that well-behaved dynamic problems can be solved with simple recursive methods, and the solution is unique and stable.